Chapter 1

Newton's Laws & Dynamical Systems

" What we know is a drop, what we don't know is an ocean." **Isaac Newton**

First Law: It states that a particle which is not acted upon by any external force, maintains its state of rest or state of uniform motion.

When Newton's law is applied to an object or system which consists of more than one particles, it is centre of mass of the object or system which obeys Newton's law. For example if two particles are placed on a smooth table at some separation, because of their natural gravitational attraction the two will move towards each other but centre of the system of two particles remains at rest.

Second Law: It states that rate of change of momentum of a system is equal to resultant external force on the system.

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{... (2.1)}$$

This form of Newton's law is valid in all cases.

If m be mass of system and $\vec{v}$ be its velocity then $\vec{p} = m\vec{v}$

Therefore, $\vec{F} = \frac{d}{dt}(m\vec{v})$

If mass of system is constant then, $\vec{F} = m\frac{d\vec{v}}{dt}$

or $$\vec{F} = m\vec{a} \quad \text{... (2.2)}$$

where $\vec{a} = \frac{d\vec{v}}{dt}$ is acceleration, evidently, equation (2.2) can be used only if mass is constant. If mass is variable, we use equation (2.1).

If F_x, F_y, F_z be Cartesian components of $\vec{F}$ and (x, y, z) be coordinate of the particle on which force is acting then equation (2.2) can be written in component form as

$$\left.\begin{aligned} F_x &= m\frac{d^2x}{dt^2} \\ F_y &= m\frac{d^2y}{dt^2} \\ F_z &= m\frac{d^2z}{dt^2} \end{aligned}\right\} \quad \text{... (2.3)}$$

However, instead of Cartesian coordinates if we use some curvilinear coordinate then we do not get similar relations. For example if F_r, F_θ, F_ϕ be components of force $\vec{F}$ in spherical polar coordinate and r, θ, ϕ be spherical polar coordinates of particle then,

$$\left.\begin{aligned} F_r &\neq m\frac{d^2r}{dt^2} \\ F_\theta &\neq m\frac{d^2\theta}{dt^2} \\ F_\phi &\neq m\frac{d^2\phi}{dt^2} \end{aligned}\right\} \qquad \ldots (3.3)$$

Thus, we cannot write Newton's law in other coordinate system by merely replacing x, y, z by corresponding coordinates.
Due to this reason we say that Newton's law is not covariant under coordinate transformation.

Third Law: It states that to every action there is equal and opposite reaction. If $\vec{F}_{12}$ be force on an object 1 due to an object 2, and F_{21} be force on 2 due to 1 then

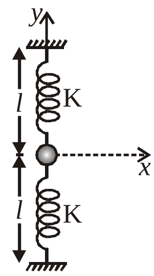
$$\vec{F}_{12} = -\vec{F}_{21}$$

Newton's third law assumes that to an action there is prompt reaction. This implies that in Newton's law it is assumed that information travels with infinite speed.

Violation of Newton's Laws: Newton's 1st and 2nd laws are strictly valid in inertial frames. In non-inertial frames Newton's laws do not hold good. In non-inertial frames we can apply Newton's law if we include pseudo forces.
There are some cases where Newton's third law also does not hold. For example magnetic force exerted by two non-parallel current carrying wires on each other are not in opposite in direction.

Example: A particle of mass *m* is connected to two springs of unstretched length '*l*' and spring constant *k* as shown in figure. Calculate acceleration of particle if it is slightly displaced along X-direction.

Soln. Let the particle is displaced by a distance *x* along +*x* direction as shown in figure.

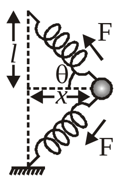


Therefore, stretched length of spring $= \sqrt{l^2 + x^2}$

elongation in springs = $\sqrt{l^2 + x^2} - l$

Restoring force on the particle due to each spring $F = k(\sqrt{l^2 + x^2} - l)$

Because of restoring force the particle moves back towards its initial position.



Equation of motion of particle $F_x = m\dfrac{d^2x}{dt^2}$

or $-2F\cos\theta = m\dfrac{d^2x}{dt^2}$ or $-2k(\sqrt{l^2+x^2} - l).\dfrac{x}{\sqrt{l^2+x^2}} = m\dfrac{d^2x}{dt^2}$

Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left(1 - \dfrac{l}{\sqrt{l^2+x^2}}\right)$ or $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1 - \left(1 + \dfrac{x^2}{l^2}\right)^{-1/2}\right]$

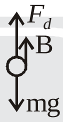
Since, $x << l,\ \left(1+\dfrac{x^2}{l^2}\right)^{-1/2} \approx 1-\dfrac{x^2}{2l^2}$

Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1-\left(1-\dfrac{x^2}{2l^2}\right)\right]$ or $\dfrac{d^2x}{dt^2} = -\dfrac{kx^3}{ml^2}$

Example: Spherical particles of a given material of density ρ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the Stoke's law, i.e, $F_d = 6\pi\eta R\mathrm{v}$, where η is the viscosity of the medium, R the radius of a particle and v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass *m* to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is

Soln. Each particle has same density but different radii and masses. We have to calculate time of fall in terms of mass of particle therefore we will write our equations explicitly in terms of mass and we will remove radius from our equations wherever it appears.

Drag force on particles



$F_d = 6\pi\eta R\mathrm{v}$, Mass of a particle $m = \rho.\dfrac{4}{3}\pi R^3$

$\therefore R = \left(\dfrac{3m}{4\pi\rho}\right)^{1/3} \qquad \therefore F_d = 6\pi\eta\left(\dfrac{3m}{4\pi\rho}\right)^{1/3} v \quad = Km^{1/3}v$, where $K = 6\pi\eta\left(\dfrac{3}{4\pi\rho}\right)^{1/3}$

If B is buoyancy force, then equation of motion of a particle is

$m\dfrac{dv}{dt} = mg - F_d - B$ where B $= \dfrac{4}{3}\pi R^3\sigma g = \dfrac{4}{3}\pi R^3\rho g.\dfrac{\sigma}{\rho} = mg\dfrac{\sigma}{\rho}$, σ = density of medium

$\therefore m\dfrac{dv}{dt} = mg - Km^{1/3}v - mg\dfrac{\sigma}{\rho}, \quad m\dfrac{dv}{dt} = mg\left(1-\dfrac{\sigma}{\rho}\right) - Km^{1/3}v$

or $\dfrac{dv}{dt} = g\left(1-\dfrac{\sigma}{\rho}\right) - Km^{-2/3}v$ ... (a)

when terminal velocity is reached $\dfrac{dv}{dt} = 0$

$\therefore 0 = g\left(1-\dfrac{\sigma}{\rho}\right) - Km^{-2/3}v_t \qquad \therefore v_t = \dfrac{g\left(1-\dfrac{\sigma}{\rho}\right)}{Km^{-2/3}}$ ...(b)

Now, if τ be the time to reach half the terminal velocity then from (a)

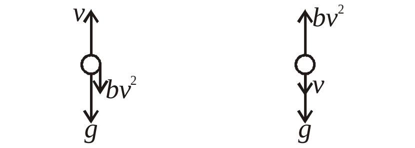
$$\int_0^{v_t/2} \frac{dv}{g\left(1-\dfrac{\sigma}{\rho}\right) - Km^{-2/3}v} = \int_0^{\tau} dt, \quad \therefore \quad -\frac{1}{Km^{-2/3}}\ln\left[\frac{g\left(1-\dfrac{\sigma}{\rho}\right) - \dfrac{Km^{-2/3}v_t}{2}}{g\left(1-\dfrac{\sigma}{\rho}\right)}\right] = \tau$$

using value of v_t from (b) we get, $\tau = \dfrac{m^{2/3}}{K}\ln 2$ or $\tau(m) = \dfrac{m^{2/3}\ln 2}{K}$

$\therefore \tau(8m)/\tau(m) = (8m)^{2/3}/m^{2/3} = 4$

Example: A particle of unit mass is thrown vertically upward with initial speed v_0. It is acted upon by a drag force bv^2 in addition to gravity where b is constant and v is instantaneous velocity of particle. Calculate speed of the particle when it returns to the point from where it was thrown.

Soln. In presence of drag force only quantity that is common in upward and downward motion is the distance covered.



upward motion   downward motion

For upward motion, initial speed = v_0,

final speed = 0, let height reached = h

equation of motion $\dfrac{dv}{dt} = -g - bv^2$

or $\dfrac{dv}{dy} \cdot \dfrac{dy}{dt} = -g - bv^3$, $\dfrac{dy}{dt} = v$ or $\dfrac{vdv}{g + bv^2} = -dy$

$$\therefore \int_{v_0}^{0} \frac{vdv}{g + bv^2} = -\int_0^h dy$$

on integration we get, $\dfrac{1}{2b} \ln\left(\dfrac{g + bv_0^2}{g}\right) = h$ ... (a)

for downward motion, initial speed = 0, final speed = v(let), height descended = h

equation of motion $\dfrac{dv}{dt} = g - bv^2$ or $\dfrac{vdv}{g - bv^2} = dy$ or $\displaystyle\int_0^v \frac{vdv}{g - bv^2} = \int_0^h dy$

on integration we get, $\dfrac{1}{2b} \ln\left(\dfrac{g}{g - bv^2}\right) = h$ ... (b)

Comparing (a) and (b) we get, $\dfrac{g + bv_0^2}{g} = \dfrac{g}{g - bv^2}$, $g - bv^2 = \dfrac{g^2}{g + bv_0^2}$

or $v = \dfrac{v_0}{\sqrt{1 + \dfrac{bv_0^2}{g}}} \Rightarrow v < v_0$

due to drag force the particle returns with a speed less than its initial speed. If drag force were absent $b = 0$, then $v = v_0$. Therefore in absence of drag force particle returns with same speed as its initial value.

Formulae of kinematics (uniformly accelerated motions): When a constant force acts on a particle its acceleration is constant. If u_x, v_x be initial and final velocity of a particle along x-direction and a_x be its acceleration along x direction then.

$$\left.\begin{aligned} v_x &= u_x + a_x t \\ v_x^2 &= u_x^2 + 2a_x x \\ x &= u_x t + \tfrac{1}{2} a_x t^2 \end{aligned}\right\} \text{if } a_x = \text{constant}$$

where x is displacement along x-direction and t is time taken. We can write similar relations for y and z direction also.

Note: If acceleration of particle is not constant then we cannot use formulae of kinematics. In that case we start with either definition of velocity or definition of acceleration i.e.

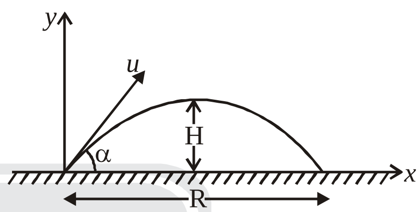
$$v_x = \frac{dx}{dt} \quad \text{or} \quad a_x = \frac{dv_x}{dt}$$

we may also have to use $a_x = \frac{F_x}{m}$ and $\frac{dv_x}{dt} = \frac{dv_x}{dx}.\frac{dx}{dt} = \frac{v_x dv_x}{dx}$

as the case may be

Projectile motion: If u be the projection speed and α be angle of projection then equation of path of projectile is,

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$



Range, $R = \frac{u^2 \sin 2\alpha}{g}$

Maximum height, $H = \frac{u^2 \sin^2\alpha}{2g}$

Total time of flight, $T = \frac{2u\sin\alpha}{g}$

Coordinate of projectile at any time $t < T$.

$$x = u\cos\alpha\, t,\ y = u\sin\alpha\, t - \frac{1}{2}gt^2$$

Note: Above results for projectile are applicable only if projectile is thrown from ground and it finally lands on ground. And projection speed is small so that projectile does not go too high. If projection speed is not small then height will be large and in that case we need to consider variation of acceleration due to gravity.

Example: A particle starts moving with acceleration $a = \alpha - \beta v$ where α and β are constants and v is its instantaneous speed. Find velocity of particle as a function of time and also find its terminal velocity.

Soln: Acceleration is variable therefore we start with definition of acceleration

$$a = \frac{dv}{dt} \quad \text{or} \quad \alpha - \beta v = \frac{dv}{dt} \quad \text{or} \quad \int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt$$

$$-\frac{1}{\beta}\ln\left(\frac{\alpha - \beta v}{\alpha}\right) = t \quad \text{or,} \quad 1 - \frac{\beta v}{\alpha} = e^{-\beta t} \qquad \therefore v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

as $t \to \infty$, $v = \frac{\alpha}{\beta}$ = constant. This is terminal velocity.

Example: A projectile is thrown at an angle α with horizontal with initial speed u after what time the projectile's velocity will be perpendicular to its initial direction.

Soln. Suppose velocity after time t becomes perpendicular to initial velocity. Velocity after time t is

$$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j} \quad = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$$

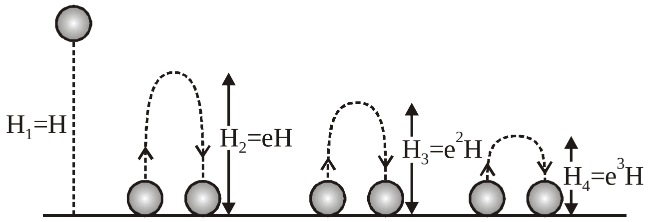
Initial velocity $\vec{u} = u(\cos\alpha\hat{i} + \sin\alpha\,\hat{j})$

$\vec{v}$ is $\perp$ to $\vec{u}$, therefore, $\vec{v}.\vec{u} = 0 \quad \therefore u\cos^2\alpha + u\sin^2\alpha - gt\sin\alpha = 0$

$\therefore t = \dfrac{u}{g\sin\alpha}$

Example: A ball is dropped from a height H. It bounces back up to a height *e* times after hitting the ground. If $e<1$, after what time the ball will finally come to rest.

Soln. Ball moves under the effect of gravity. Therefore magnitude of acceleration of ball during upward or downward movement remains constant.



For first bounce

$y = H,\ u_y = 0,\ a_y = g$

$y = u_y t + \frac{1}{2}a_y t^2$

$\therefore H = 0 + \frac{1}{2}gt^2 \quad \therefore t = \sqrt{\frac{2H}{g}}$

Therefore time of fall before first bounce $t_1 = \sqrt{\frac{2H}{g}}$

Under gravity, time for going up a certain distance is equal to time for going down. Therefore time interval between first bounce and second bounce is

$t_2 = 2\sqrt{\frac{2H_2}{g}} = 2\sqrt{\frac{2eH}{g}}$

Similarly $t_3 = 2\sqrt{\frac{2e^2H}{g}},\ t_4 = 2\sqrt{\frac{2e^3H}{g}},.....$

Therefore, total time elapsed before ball stops is

$T = t_1 + t_2 + t_3 + ...$

$$= \sqrt{\frac{2H}{g}}\left[1 + 2\left(e^{1/2} + e^{2/2} + e^{3/2} + ...\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2\frac{e^{1/2}}{1-e^{1/2}}\right] = \sqrt{\frac{2H}{g}}\left(\frac{1+\sqrt{e}}{1-\sqrt{e}}\right)$$

Some Important Terms:

Conservative force: A force is said to be conservative if curl of the force is zero.

i.e. if $\vec{\nabla}\times\vec{F} = 0$ then, $\vec{F}$ is conservative force.

Work done by a conservative force is independent of path followed and work done along a closed path is zero. Gravitational and electrostatic forces are conservative. Friction, viscous forces are not conservative.

Potential energy: It is defined as work done by conservative force in taking the system from given configuration (position) to a reference position. Reference positions is that where the conservative force vanishes.

Potential energy can be directly obtained from relation

$U = -\int F dr$ or $U = -\int F dx$

for attractive force F is used with negative sign and for repulsive force F is used with positive sign.

Spring Force: When a spring is either stretched or compressed such that its length changes by x then restoring force or tension in the spring is $-kx$.