Chapter 2

Stability Analysis

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

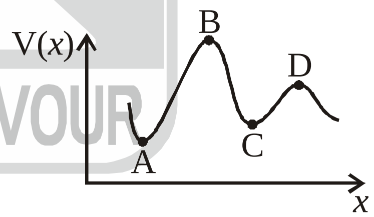
Equilibrium criteria in one dimension:

If $V(x)$ be the potential under which a particle is moving then force acting on the particle is $F_x = -\frac{dV(x)}{dx}$

at equilibrium point $F_x = 0$. Therefore, condition for equilibrium in terms of potential becomes, $\frac{dV(x)}{dx} = 0$

In $V(x)$ versus x graph $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x axis. Therefore in the figure shown below point A, B, C and D are equilibrium points.

Stable equilibrium point: A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice versa. That is the force tries to bring it back. Therefore at stable equilibrium point $\frac{dF_x}{dx} < 0$



$\therefore \frac{d^2V(x)}{dx^2} > 0$ (condition for minimum)

Thus, a stable equilibrium point is a minimum on $V(x)$ versus x plot.
Therefore points A and C in the figure shown above are stable point.

Unstable equilibrium point: In this case a particle at equilibrium point is when displaced towards right it experiences force also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point, $\frac{dF_x}{dx} > 0$

$\therefore \frac{d^2V(x)}{dx^2} < 0$ (condition of maximum)

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points B and D in the figure shown above are unstable points.

Frequency of oscillation: We know a particle of mass *m* moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point. For a particle moving under some arbitrary potential $V(x)$, if x_0 be the stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x) = V(x_0) + (x - x_0)\left.\frac{dV}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{\underline{|3}}\left.\frac{d^3V}{dx^3}\right|_{x=x_0} + \ldots\ldots$$

where $(x - x_0)$ is displacement from stable equilibrium point.

At this point, $\left.\frac{dV}{dx}\right|_{x=x_0} = 0$, therefore, if $(x - x_0)$ be small then

$$V(x) = V(x_0) + \frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} \quad \text{(higher power terms neglected)}$$

or $\quad V(x) = V(x_0) + \frac{1}{2}k(x - x_0)^2$

Therefore frequency of oscillation about stable equilibrium is

$$\omega = \sqrt{\frac{k}{m}} \quad \text{where, } k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}, \ k \text{ is called force constant.}$$

Equilibrium criteria in two dimension: If a particle is moving under a two dimensional potential $V(x, y)$ then, at equilibrium point,

$$\left.\frac{\partial}{\partial x}V(x,y)\right|_{x_0,y_0} = 0, \quad \left.\frac{\partial}{\partial y}V(x,y)\right|_{x_0,y_0} = 0$$

for stable equilibrium point (minimum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} > 0, \quad \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} > 0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} \cdot \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} > \left[\left.\frac{\partial^2}{\partial x \partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

For unstable equilibrium point (maximum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} < 0, \quad \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} < 0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} \cdot \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} > \left[\left.\frac{\partial^2}{\partial x \partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

for saddle point:

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}<\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

Example: A particle of mass m is moving under a one dimensional potential $V(x)=-ax+bx^2$ where $a>0$ and $b>0$. Find the equilibrium points and find frequency of oscillation about the stable equilibrium.

Soln. At equilibrium point, $\left.\frac{dV}{dx}\right|_{x=x_0}=0$

$$\therefore\ -a+2bx_0=0 \quad \text{or} \quad x_0=\frac{a}{2b}$$

Thus, $x_0=\frac{a}{2b}$ is an equilibrium point.

To know whether it is stable or unstable equilibrium point let us calculate second derivative of potential at equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=2b>0$$

Therefore $x_0=\frac{a}{2b}$ is stable equilibrium point, force constant $k=\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=2b$

Therefore, frequency of oscillation is

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{2b}{m}}$$

Example: Potential corresponding to force between the atoms of a diatomic molecule is $V(r)=\frac{a}{r^{12}}-\frac{b}{r^6}$ where a and b are positive constants and r is separation between the atoms. Calculate bond length for stable configuration and also calculate frequency of oscillation of atoms if mass of each atom be m.

Soln. For stable configuration $\left.\frac{dV}{dr}\right|_{r=r_0}=0$

$$-\frac{12a}{r_0^{13}}+\frac{6b}{r_0^7}=0\Rightarrow r_0=\left(\frac{2a}{b}\right)^{1/6}$$

Therefore bond length for stable configuration is $\left(\frac{2a}{b}\right)^{1/6}$

Force constant $k=\left.\frac{d^2V}{dr^2}\right|_{r=r_0}=\frac{12\times13a}{r_0^{14}}-\frac{6\times7b}{r_0^8}=\frac{1}{r_0^8}\left(\frac{12\times13a}{r_0^6}-6\times7b\right)$

$$=\left(\frac{b}{2a}\right)^{8/6}\left(\frac{12\times13a}{2a/b}-42b\right)=\left(\frac{b}{2a}\right)^{4/3}.36b=\frac{18}{2^{1/3}}.\frac{b^{7/3}}{a^{4/3}}$$

Reduced mass of system $\mu = \frac{m.m}{m+m} = m/2$

Frequency of oscillation $\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{18}{2^{1/3}} \cdot \frac{b^{7/3}}{m/2\, a^{4/3}}} \; = 6\left(\frac{b^7}{2m^3\, a^4}\right)^{1/6}$

Example: A particle of mass '*m*' is moving under potential $V(x) = ax^3 - bx^2$. Initially the particle is at rest at stable point. What minimum speed be given to the particle so that it reaches unstable point. Plot the potential versus *x*.

Soln. For equilibrium point $\left.\frac{dV}{dx}\right|_{x=x_0} = 0$

$$3ax_0^2 - 2bx_0 = 0 \;\Rightarrow x_0 = 0, \frac{2b}{3a}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = 6ax_0 - 2b\,,\quad \left.\frac{d^2V}{dx^2}\right|_{x=0} = -2b < 0,\ \therefore\ x_0 = 0 \text{ is unstable point}$$

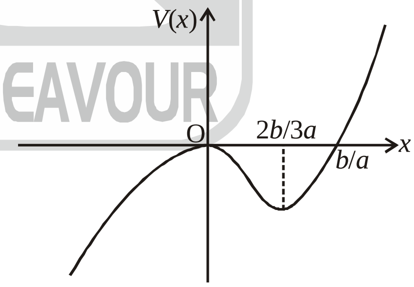
$$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2b}{3a}} = 2b > 0,\ \therefore\ x_0 = \frac{2b}{3a} \text{ is stable point}$$

To calculate speed let us apply conservation of energy.
Total energy initial = Total energy final

(Kinetic Energy + Potential Energy) at $x = \frac{2b}{a}$ = (Kinetic Energy + Potential Energy) at $x = 0$

for minimum speed (u) at stable point the particle will just reach unstable point and stops there.

$$\therefore \frac{1}{2}mu^2 + V\left(x = \frac{2b}{3a}\right) = \frac{1}{2}m.0^2 + V(x=0)$$



$$\frac{1}{2}mu^2 + a\left(\frac{2b}{3a}\right)^3 - b\left(\frac{2b}{3a}\right)^2 = 0 + 0$$

$$u^2 = -\frac{2}{m}\left(\frac{2b}{3a}\right)^2\left(\frac{2b}{3} - b\right) = \frac{2}{m} \cdot \frac{4b^2}{9a^2} \cdot \frac{b}{3} = \frac{8b^3}{27ma^2}$$

$$\therefore u = \sqrt{\frac{8b^3}{27ma^2}}$$

Example : A particle is moving under potential $V(r) = \frac{a}{r^2} - \frac{b}{r}$. Calculate the minimum value of potential energy.

Soln. For potential to be minimum $\left.\frac{dV}{dr}\right|_{r=r_0} = 0$

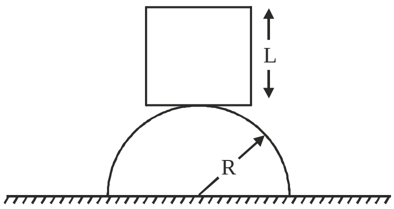
$$-\frac{2a}{r_0^3} + \frac{b}{r_0^2} = 0 \qquad \therefore\ r_0 = \frac{2a}{b}$$

$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0} = \frac{6a}{r_0^4} - \frac{2b}{r_0^3} = \frac{1}{r_0^3}\left(\frac{6a}{r_0} - 2b\right) = \frac{1}{r_0^3}\left(\frac{6a}{2a/b} - 2b\right) = \frac{b}{r_0^3} > 0$$

Therefore at $r = r_0$ potential is minimum

$$\therefore \qquad V_{\min} = V(r_0) = \frac{1}{r_0}\left(\frac{a}{r_0} - b\right) = -\frac{b^2}{4a}$$

Example: A cube is placed on the top of a fixed hemisphere as shown in figure. What should be relation between length of side of cube and radius of hemisphere so that cube has stable equilibrium.



Soln. To discuss equilibrium of cube we first write its potential energy as function of angle from its equilibrium position. As shown in the figure below, initially point A was in contact with spherical surface but now in displaced position point B is in contact. Therefore,

$AB = R\theta$

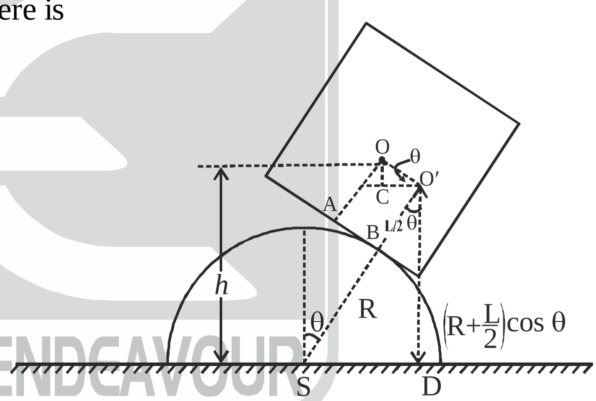
Height of centre of cube from centre level of hemisphere is

$h = OC + O'D$

$= OO'\sin\theta + O'S\cos\theta$

$= AB\sin\theta + (O'B + BS)\cos\theta$

$= R\theta\sin\theta + \left(\frac{L}{2} + R\right)\cos\theta$



Potential energy of the cube

$$V(\theta) = Mgh \quad = Mg\left[R\theta\sin\theta + \left(\frac{L}{2} + R\right)\cos\theta\right]$$

$\theta = 0$ is equilibrium position of the cube. For this position to be stable equilibrium position, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0} > 0$

$$\therefore \frac{d^2}{d\theta^2} Mg\left[R\theta\sin\theta + \left(\frac{L}{2} + R\right)\cos\theta\right]_{\theta=0} > 0$$

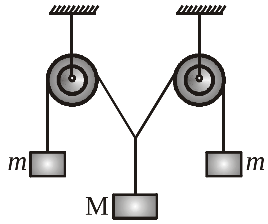
or $\frac{d}{d\theta}\left[R\theta\cos\theta + R\sin\theta - \left(\frac{L}{2} + R\right)\sin\theta\right]_{\theta=0} > 0$

or $\left[R\cos\theta - R\theta\sin\theta + R\cos\theta - \left(\frac{L}{2} + R\right)\cos\theta\right]_{\theta=0} > 0$

or $\left[2R - \left(\frac{L}{2} + R\right)\right] > 0 \quad \therefore R - \frac{L}{2} > 0$ or $2R > L$

Thus, cube can be in stable equilibrium position if its side length is less than diameter of hemisphere.

Example: For the mass pulley system shown in figure, what should be relation between m and M so that system remains in stable equilibrium position. Pulley are smooth and strings are tight and inextensible.

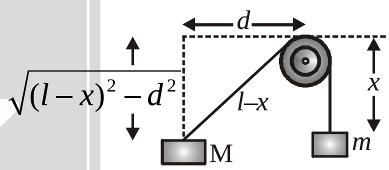


Soln. The pulleys are fixed. Therefore we can write potential energy of the system by specifying position of blocks with respect to pulleys.

Let 'l' be length of string and 'd' be the half distance between two pulleys. Therefore, l and d are constants. If x be distance of m below the pulley as shown in figure then potential energy of system is

$$V(x) = -2mgx - Mg\sqrt{(l-x)^2 - d^2} \qquad \therefore \frac{dV}{dx} = -2mg + \frac{Mg(l-x)}{\sqrt{(l-x)^2 - d^2}}$$

for equilibrium $\left.\dfrac{dV}{dx}\right|_{x=x_0} = 0 \qquad \therefore -2mg + \dfrac{Mg(l-x_0)}{\sqrt{(l-x_0)^2 - d^2}} = 0$



or $\dfrac{2m}{M} = \dfrac{l-x_0}{\sqrt{(l-x_0)^2 - d^2}}$ ... (a)

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = \frac{Mgd^2}{\left[(l-x_0)^2 - d^2\right]^{3/2}} > 0 \text{ for all values of } d > 0$$

since, $\dfrac{l-x_0}{\sqrt{(l-x_0)^2 - d^2}} > 1 \qquad \therefore$ from (a) $\dfrac{2m}{M} > 1$ or $2m > M$

SOLVED EXAMPLES

1. The potential energy between two atoms are given $v(r) = \dfrac{a}{r^{12}} - \dfrac{b}{r^6}$

where a, b positive constants.

(i) Find the equilibrium distance of two atoms.

(ii) Plot the potential

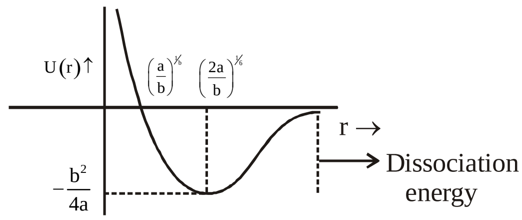
(iii) Calculate the frequency of small oscillation.

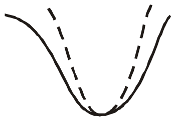
Soln. (i) For equilibrium potential energy should be minimum.

$$\frac{dU}{dr} = -\frac{12a}{r^{13}} + \frac{6b}{r^7} = 0 \Rightarrow r^6 = \frac{2a}{b}$$

$$r = \left(\frac{20}{b}\right)^{1/6} = r_0 \qquad \text{[Position of stability equilibrium]}$$

$$U(r_0) = \frac{ab^2}{4a^2} - \frac{b.b}{2a} = \frac{b^2}{4a} - \frac{b^2}{2a} = -\frac{b^2}{4a}$$

(ii) 

(iii)  Approximate to harmonic oscillator

$$U(r_0)=\frac{-b^2}{4a};\ U(r)=U(r_0)+(r-r_0)\left.\frac{dU}{dr}\right|_{r_0}+\frac{(r-r_0)^2}{2}\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

$$\frac{d^2U}{dr^2}=\frac{12\times 13a}{r^{14}}-\frac{42b}{r^8};\ \left.\frac{d^2U}{dr^2}\right|_{r_0}=\frac{156a}{r^{14}}-\frac{42b}{r^8}=\frac{156a}{(2a)^{14/6}}-\frac{42b}{\left(\frac{2a}{h}\right)^{8/6}}$$

$$=\frac{156\,a}{\left(\frac{2a}{b}\right)^{7/3}}-\frac{42b}{\left(\frac{2a}{b}\right)^{4/3}}=\text{constant}=c$$

(Force constant equivalent to spring constant

$$U(r)=U(r_0)+\frac{c}{2}(r-r_0)^2$$

So, the force $=-\frac{dU}{dr}=-\frac{c}{2}.2(r-r_0)^2=-c(r-r_0)$

Force $=-\nabla U;\ U=U(r)$ only

$\vec{F}=m\,\vec{a}$ (Here introduced to harmonic oscillator may necessary).

Equation of motion, $m\frac{d^2r}{dt^2}=-c(r-r_0);\ \frac{d^2r}{dt^2}+\frac{c}{m}(r-r_0)=0$

Take $c=m\omega^2;\ \omega=\sqrt{\frac{c}{m}}$

2. A particle of mass 'm' moving in a potential

$V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$ $\quad(\omega_0\ \&\ \text{are constant})$

Find the angular frequency of small oscillation.

$$\frac{dv}{dx}=m\omega_0^2x-\frac{a}{m\,x^3}=0$$

$$m\omega_0^2x_0-\frac{a}{mx^3}=0;\ x_0^4=\frac{a}{m^2\omega_0^2};\ \chi_0=\left(\frac{a}{m^2\omega_0^2}\right)^{1/4}$$

Equilibrium distance, $\frac{d^2z}{dx^2}=m\omega_0^2+\frac{3a}{mx^4}$

$$\left.\frac{d^2r}{dx^2}\right|_{x=x_0} = m\omega_0^2 + \frac{3a}{ma}m^2\omega^2 = 4m\omega_0^2 \qquad \left(x_0^4 = \frac{a}{m^2\omega_0^2}\right)$$

$$U(x) = U(x_0) + (x - x_0)\left.\frac{dU}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$$

$$= U(x_0) + (x-x_0)^2\, 2m\omega_0^2$$

$$\text{Force} = -\frac{dU}{dx} = -4m\omega_0^2(x - x_0)$$

Equation of motion, $m\frac{d^2x}{dt^2} = F = -4\omega_0^2(x - x_0)$

$$\frac{d^2x}{dt^2} + 4\omega_0^2(x - x_0) = 0$$

Hence, the frequency of small oscillation, $\omega = \sqrt{4\omega_0^2} = 2\omega_0$.

3. A particle of mass '*m*' is constrained to move in one dimension under a potential $u(x) = \frac{\alpha}{x^2} - \frac{\beta}{x}$, where α and β are positive constants. Show that period of small oscillations about the equilibrium point is, $T = 4\pi\sqrt{\frac{2\alpha^3 m}{\beta^4}}$.

Soln. At equilibrium point $(x = x_0)$

$$\left.\frac{du}{dx}\right|_{x=x_0} = 0$$

$$-\frac{2\alpha}{x_0^3} + \frac{\beta}{x_0^2} = 0 \qquad \therefore\ x_0 = \frac{2\alpha}{\beta}$$

time period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where k is force constant and is given by

$$k = \left.\frac{d^2u}{dx^2}\right|_{x=x_0} = \frac{6\alpha}{x_0^4} - \frac{2\beta}{x_0^3} = 6\alpha\cdot\frac{\beta^4}{16\alpha^4} - \frac{2\beta^4}{8\alpha^3}$$

$$= \frac{3}{8}\cdot\frac{\beta^4}{\alpha^3} - \frac{2}{8}\cdot\frac{\beta^4}{\alpha^3} \quad \Rightarrow k = \frac{\beta^4}{8\alpha^3}$$

$$\therefore\ T = 2\pi\sqrt{\frac{8\alpha^3}{\beta^4}\cdot m} = 4\pi\sqrt{\frac{2\alpha^3 m}{\beta^4}}$$

4. A particle of unit mass moves in a potential $V(x)=ax^2+\frac{b}{x^2}$, where a and b are positive constants. The angular frequency of small oscillations about the minimum of the potential is: **[NET June 2011]**

(a) $\sqrt{8b}$ (b) $\sqrt{8a}$ (c) $\sqrt{8a/b}$ (d) $\sqrt{8b/a}$

Soln. $V(x)=ax^2+\frac{b}{x^2}$

Angular frequency, $\omega=\sqrt{\frac{k}{m}}$, where, $k=\left.\frac{d^2V}{dx^2}\right|_{x=x_0}$

Where, x_0 is the point at which $V(x)$ is minimum.

For minimum, $\left.\frac{dV}{dx}\right|_{x=x_0}=0 \Rightarrow 2ax_0-\frac{2b}{x_0^3}=0 \Rightarrow x_0^2=\sqrt{\frac{b}{a}}$

So, $k=2a+\frac{6b}{x_0^4}=8a$ and $\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{8a}{1}}=\sqrt{8a}$