Chapter 5

COMPLEX ANALYSIS

5.1 Basic Review of Complex Numbers

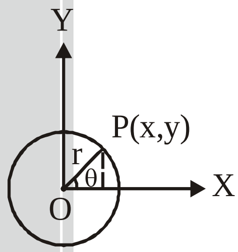
Various representation of Complex number :

A complex number (z) is represented as $z = x + iy$, where x is the real part and y is imaginary part.
The conjugate of complex number z is represented by $\bar{z} = x - iy$

The modulus of complex number $z = |z| = r = \sqrt{x^2 + y^2}$ and argument of $z = Argz = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

The complex number (z) in polar form (r, θ) can also be written as

$$z = x + iy = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$$



where $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2 = |z|^2$

This implies that $|z| = r$ represents a circle centred at the origin and having radius r.

Properties of modulus of z:

If z_1 and z_2 are two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ (ii) $|z_1 - z_2| \ge |z_1| - |z_2|$

(iii) $|z_1 z_2| = |z_1||z_2|$ (iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Properties of argument of z:

(i) $Arg(z_1.z_2.z_3.....z_n) = Arg.(z_1) + Arg.(z_2) + Arg.(z_3) + ... + Arg(z_n)$

(ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

Some important Relations:

(i) $e^{i\theta} = \cos\theta + i\sin\theta$, (ii) $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, (iii) $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$

(iv) $\cos h\theta = \frac{(e^{\theta} + e^{-\theta})}{2}$, (v) $\sin h\theta = \frac{(e^{\theta} - e^{-\theta})}{2}$

De-Moiver's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Complex cube roots of unity:

$$x^3 = 1 \Rightarrow x = 1, \omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2} \text{ such that } 1+\omega+\omega^2 = 0 \text{ and } \omega^3 = 1$$

Example 1: Multiplying a complex number z by $1 + i$ rotates the radius vector of z by an angle of

(a) 90º clockwise
(b) 45º anticlockwise
(c) 45º clockwise
(d) 90º anticlockwise

Soln: $z = r\,e^{i\theta} \Rightarrow (1+i)z = \left(\sqrt{2}\,e^{i\pi/4}\right) r\,e^{i\theta} = \sqrt{2}\,r\,e^{i(\theta+\pi/4)}$

Multiplying (1 + i) with z rotates the radius vector anticlockwise by 45º and increases the modulus by a factor of $\sqrt{2}$.

Correct option is (a)

Example 2: If $z = (\lambda+3) + i\sqrt{5-\lambda^2}$ (λ is a real parameter and $i = \sqrt{-1}$), then the locus of z will be

(a) circle
(b) ellipse
(c) parabola
(d) hyperbola

Soln: $z = (\lambda+3) + i\sqrt{5-\lambda^2} \Rightarrow$ real part $x = \lambda+3$ and imaginary part $y = \sqrt{5-\lambda^2}$

$\Rightarrow y^2 = 5-\lambda^2 = 5-(x-3)^2 \Rightarrow (x-3)^2 + y^2 = 5$ (Equation of circle)

Correct option is (a)

5.2 Function of A complex Variable

Basic Representation:

$$f(z) = u(x,y) + iv(x,y)$$

Example: $f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)^2}_{u(x,y)\text{ is real part}} + \underbrace{i2xy}_{v(x,y)\text{ is imaginary part}}$

Existance of $\lim_{z\to z_0} f(z)$**:**

The limit will exists only if the limiting value is independent of the path along which z approaches z_0

Example 3: Find whether the limit $\lim_{z\to 0}\frac{z}{|z|}$ exist or not.

Soln: $z \to 0$ means $x \to 0$ & $y \to 0$

For z = 0, we have to choose a path passing through a origin.

Therefore, we have choosen a straight line passing through the origin i.e. y = mx

$$\lim_{z\to 0}\frac{z}{|z|} = \lim_{\substack{z\to 0\\ y\to 0}}\frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\frac{x+imx}{\sqrt{x^2+m^2x^2}} = \frac{1+im}{\sqrt{1+m^2}}$$

Therefore, the limit depends on m i.e. slope of the straight line. Thus, the limiting values is dependent on the path and the limit does not exists.

Example 4: Calculate the value $\lim_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$

Soln: $\lim_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}=\lim_{z\to\infty}\frac{z^3\left(i+\frac{i}{z^2}-\frac{1}{z^3}\right)}{z\left(2+\frac{3i}{z}\right)z^2\left(1-\frac{1}{z}\right)^2}=\frac{i}{2}$

Differentiability of complex function:

$$f'(z)=\underset{\delta z\to 0}{Lt}\frac{[f(z+\delta z)-f(z)]}{\delta z}$$

The function will be differentiable if limit exists i.e the limiting value will be independent of path along with $\delta z\to 0$.

Example: $f(z) = (4x+y)+i(4y-x) \Rightarrow u=(4x+y)$ and $v=(4y-x)$

$\Rightarrow f(z+\delta z) = 4(x+\delta x)+(y+\delta y)+i\,[4(y+\delta y)-(x+\delta x)]$

$\Rightarrow f(z+\delta z)-f(z) = 4\delta x+\delta y+i(4\delta y-\delta x)$

$$\Rightarrow \frac{f(z+\delta z)-f(z)}{\delta z}=\frac{4\delta x+\delta y+i(4\delta y-\delta x)}{\delta z}\Rightarrow\frac{\delta f}{\delta z}=\frac{4\delta x+\delta y-i\delta x+4i\delta y}{\delta x+i\delta y}$$

Along real axis : $\delta x=\delta z$, $\delta y=0$, $\Rightarrow \frac{\delta f}{\delta z}=4-i$

Along imaginary axis : $i\delta y=\delta z$, $\delta x=0$, $\Rightarrow \frac{\delta f}{\delta z}=4-i$

Along a line : $y=x$, $\delta y=\delta x$, $\delta z=(1+i)\,\delta x$, $\Rightarrow \frac{\delta f}{\delta z}=\frac{5\delta x+3i\delta x}{(1+i)\delta x}=\frac{5+3i}{1+i}=4-i$

Therefore, $f(z) = (4x+y)+i(4y-x)$ will be differentiable.

5.3 Complex Analytic Function

A function $f(z)$ is said to be analytic at a point $z=z_0$ if it is single valued and has the derivative at every point in some neighhourhood of z_0. The function $f(z)$ is said to be analytic in a domain D if it is single valued and is differentiable at every point of domain D.

Cauchy Reamann Equations:

For a function $f(z)=u+iv$ to be analytic at all points in some region R,

Necessary conditions:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{ and }\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Sufficient Condition: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y.

Derivative of $f(z)$**:** $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Example 5: Check whether $f(z)$ = sinz is analytic or not.

Soln: $f(z) = \sin z = \sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy) = \sin x.\cosh y + i\cos x.\sinh y$

Therefore, u = sinx.coshy and v = cosx.sinhy

$\frac{\partial u}{\partial x} = \cos x.\cosh y; \frac{\partial u}{\partial y} = \sin x.\sinh y; \frac{\partial v}{\partial x} = -\sin x.\sinh y; \frac{\partial v}{\partial y} = \cos x.\cosh y$

So, C-R equation is satisfied, given f(z) is analytic.

Example 6: Given an analytic function

$$f(x,y) = \phi(x,y) + i\psi(x,y) \text{ where } \phi(x,y) = x^2 + 4x - y^2 + 2y.$$

If C is a constant, then which of the following relations is true? **[JEST 2015]**

(a) $\psi(x,y) = x^2y + 4y + C$ (b) $\psi(x,y) = 2xy - 2x + C$

(c) $\psi(x,y) = 2xy + 4y - 2x + C$ (d) $\psi(x,y) = x^2y - 2x + C$

Soln: The condition of analytic function are

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \qquad \Bigg| \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

We have, $\phi(x,y) = x^2 + 4x - y^2 + 2y$

Therefore, $\frac{\partial \phi}{\partial x} = 2x + 4$ And $\frac{\partial \phi}{\partial y} = -2y + 2$

$\Rightarrow \quad \frac{\partial \psi}{\partial y} = 2x + 4 \qquad \Rightarrow \quad -\frac{\partial \psi}{\partial x} = -2y + 2$

$\Rightarrow \quad \psi = 2xy + 4y + c(x) \qquad \Rightarrow \quad \psi = 2xy - 2x + c(y)$

Therefore, $\psi(x,y) = 2xy + 4y - 2x + c(x,y)$

Correct option is (c)

Example 7: If $f(x,y) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 + 2iy(1-x-ax)$ is a complex analytic function then find the value of a. **[TIFR 2013]**

Soln: $u(x,y) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 \qquad \Rightarrow \frac{\partial u}{\partial x} = 2x + 2 + 2ax$

$v(x,y) = 2y(1-x-ax) \qquad \Rightarrow \frac{\partial v}{\partial y} = 2(1-x-ax)$

According to Cauchy Reamann equation, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \Rightarrow 4x=-4ax \Rightarrow a=-1$

Example 8: The harmonic conjugate function of $u(x,y)=2x(1-y)$ corresponding to a complex analytic function $\omega=u(x,y)+iv(x,y)$ is given $v(x,y)=\alpha x^2+\beta y+\gamma y^2$ (Taking the integration constant to be zero). Which of the following statement is true ?

(a) $\alpha-\gamma=\beta$ (b) $\alpha+\gamma+\beta=0$ (c) $\alpha+\gamma=\beta$ (d) $\alpha\gamma\beta=1$

Soln: $u(x,y)=2x(1-y)$

$$\Rightarrow \frac{\partial u}{\partial x}=2(1-y)=\frac{\partial v}{\partial y} \Rightarrow v=2y-y^2+f_1(x)$$

$$\Rightarrow \frac{\partial u}{\partial y}=-2x=-\frac{\partial v}{\partial x} \Rightarrow v=x^2+f_2(y)$$

Therefore, the imaginary part of the complex function $v=x^2-y^2+2y$

Comparing with the question, $\alpha=1, \beta=2, \gamma=-1 \quad \Rightarrow \alpha-\gamma=\beta$

Correct answer is (a)

Cauchy Reamann equations in Polar co-ordiantes:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Derivative of $f(z)$: $f'(z)=(\cos\theta-i\sin\theta)\frac{\partial f}{\partial r}=-\frac{i}{r}(\cos\theta-i\sin\theta)\frac{\partial f}{\partial \theta}$

Harmonic Function

Any function which satisfies the Laplace's equation, is known as harmonic function.
If u + iv is an analytic function, then u, v are conjugate harmonic functions i.e.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ and } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Example 9: Find the values of m, n such that $f(x,y)=x^2+mxy+ny^2$ is harmonic in nature.

Soln: Since, $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{dy^2}=0 \Rightarrow 2n+2=0 \Rightarrow n=-1$; '$m$' can take any value.

Method for finding conjugate Function:

Case 1: $f(z)=u+iv$, and u is known.

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy \Rightarrow v=-\int\frac{\partial u}{\partial y}dx+\int\frac{\partial u}{\partial x}dy$$

Case 2: $f(z)=u+iv$, and v is known

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\frac{\partial v}{\partial y}dx-\frac{\partial v}{\partial x}dy \Rightarrow u=\int\frac{\partial v}{\partial y}dx-\int\frac{\partial v}{\partial x}dy$$

Example 10: Find the imaginary part of the complex analytic function whose real part is $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Soln: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y} \Rightarrow v = 3x^2y - y^3 + 6xy + f_1(x)$

$\frac{\partial u}{\partial y} = -6xy - 6y = -\frac{\partial v}{\partial x} \Rightarrow v = 3x^2y + 6xy + f_2(y)$

$v(x,y) = 3x^2y - y^3 + 6xy + C$

Milne-Thomson Method : (To find Analytic function if either 'u' or 'v' is given)

Case 1: When 'u' is given,

(1) Find $\frac{\partial u}{\partial x} = \phi_1(x,y)$ and $\frac{\partial u}{\partial y} = \phi_2(x,y)$

(2) Replace x by z and y by 0 in $\phi_1(x, y)$ and $\phi_2(x, y)$ to get $\phi_1(z, 0)$ and $\phi_2(z, 0)$.

(3) Find $f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\}dz + c$

Case 2: When 'v' is given,

(1) Find $\frac{\partial v}{\partial x} = \psi_2(x,y)$ and $\frac{\partial v}{\partial y} = \psi_1(x,y)$

(2) Replace x by z and y by 0 in $\psi_1(x,y)$ and $\psi_2(x,y)$ to get $\psi_1(z,0)$ and $\psi_1(z,0)$.

(3) Find f(z) = $\int \{\psi_1(z,0) + i\psi_2(z,0)\}dz + c$

Example 11: Find the analytical function whose imaginary part is $v(x,y) = e^x(x\cos y - y\sin y)$

Soln: $\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \Psi_2(x,y)$ $\Rightarrow \frac{\partial v}{\partial y} = -e^x x\sin y - e^x(\sin y + y\cos y) = \Psi_1(x,y)$

$\Psi_1(z,0) = 0$ and $\Psi_1(z,0) = e^z z + e^z$ $\Rightarrow f(z) = \int 0 + i\left[e^z z + e^z\right]dz = ize^z + C$

Example 12: If the real part of a complex analytic function f(z) is given as, $u(x,y) = e^{-2xy}\sin(x^2 - y^2)$, then f(z) can be written as

(a) $ie^{iz^2} + C$ (b) $-ie^{iz^2} + C$ (c) $-ie^{-iz^2} + C$ (d) $ie^{-iz^2} + C$

Soln: $u(x,y) = e^{-2xy}\sin(x^2 - y^2)$

$\Rightarrow \frac{\partial u}{\partial x} = e^{-2xy}(-2y)\sin(x^2 - y^2) + e^{-2xy}\cos(x^2 - y^2)2x = \phi_1(x,y)$

$\Rightarrow \frac{\partial u}{\partial y} = e^{-2xy}(-2x)\sin(x^2 - y^2) + e^{-2xy}\cos(x^2 - y^2)(-2y) = \phi_2(x,y)$

$\therefore \quad \phi_1(z,0) = \cos z^2 .2z$, $\phi_2(z,0) = \sin z^2(-2z)$

$\therefore \quad f(z) = \int \left(\cos z^2 .2z - i\sin z^2.(-2z)\right)dz + c = 2\int\left(\cos z^2 + i\sin z^2\right).zdz + c$

$= 2\int e^{iz^2}.zdz + c = -i\, e^{iz^2} + c$

Correct option is (b)

5.4 Power Series Expansion of Complex Function

Every analytic function which is analytic at $z = z_0$ can be expanded into power series about $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

where, z_0 is the centre of power series.

For every power series there are three possibilities regarding the region of convergence of power series.

(i) The series converges within a disc

Example : $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ...$

Here $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1}}{z^n}\right| = |z|$; so for $|z| < 1$ the series will converge i.e. it will converge within the circle centered at the origin and of radius 1 unit.

(ii) The power series converges in the whole complex plane.

Example : $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$

Here, $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1}}{(n+1)!}\frac{n!}{z^n}\right| = \frac{|z|}{n+1} \to 0$ as $n \to \infty$ for all values of z i.e. the series will converge in the entire complex plane.

(iii) The power series converges for a particular values of z.

Example: $\sum_{n=0}^{\infty} n!z^n = 1 + z + 2z^2 + 6z^3 + ...$

Here, $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{(n+1)!\,z^{n+1}}{n!\,z^n}\right| = (n+1)|z| \to \infty$ as $n \to \infty$ for all values of z except z = 0 i.e. the series will converge only for z = 0.

Radius of convergence of power series:

Consider a circle centered at $z = z_0$ and radius r i.e. $|z - z_0| = R$, such that the power series is convergent for all points of the region $|z - z_0| < R$ (i.e. within the circle) and divergent for all points of the region $|z - z_0| > R$ (outside the circle). Therefore, R is said to be the radius of convergence of power series and defined as

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Example 13: Calculate the radius of convergence and region of convergence for the following power series

$$\sum_{n=0}^{\infty}\frac{(2n+3)}{(2n+5)(n+5)}z^n$$

Soln: $R = \lim_{n\to\infty}\left|\frac{2n+3}{(2n+5)(n+5)}\frac{(2(n+1)+5)(n+1+5)}{2(n+1)+3}\right| = \lim_{n\to\infty}\left|\frac{\left(2+\frac{3}{n}\right)\left(2+\frac{7}{n}\right)\left(1+\frac{6}{n}\right)}{\left(2+\frac{5}{n}\right)\left(1+\frac{5}{n}\right)\left(2+\frac{5}{n}\right)}\right| = 1$

Region of convergence $|z| < 1$

This series is convergent within a circle of radius 1 and centre (0, 0).

Example 14: Prove that the series

$$1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2+.....$$

has unit radius of convergence.

Soln: Neglecting the first term,

$$a_n = \frac{a(a+1)..(a+n-1)b(b+1)...(b+n-1)}{1.2...n.c(c+1)....(c+n-1)}$$

$$a_{n+1} = \frac{a(a+1)...(a+n-1)(a+n)(b+1)....(b+n-1)(b+n)}{1.2...n(n+1)c(c+1)....(c+n-1)(c+n)}$$

Dividing, $\frac{a_{n+1}}{a_n} = \frac{(n+a)(n+b)}{(n+1)(c+n)} = \frac{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(1+0)(1+0)}{(1+0)(1+0)} = 1 \text{ or } R = 1$$

Example 15: Find the radius of convergence of the series

$$\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+......$$

Soln: The coefficient of z^n of the given power series is given by

$$a_n = \frac{1.3.5...(2n-1)}{2.5.8...(3n-1)}$$

$$a_{n+1} = \frac{1.3.5...(2n-1)(2n+1)}{25.8...(3n-1)(3n+2)}$$

So, $$\frac{a_{n+1}}{a_n} = \frac{2n+1}{3n+2} = \frac{2}{3} \cdot \frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$$

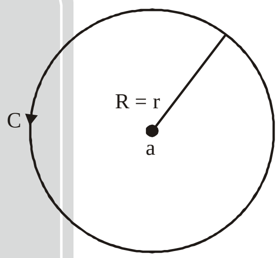
Therefore, $$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3} \cdot \frac{(1+0)}{(1+0)} = \frac{2}{3} \Rightarrow R = \frac{3}{2}$$

Taylor Series Expansion

If a function f(z) is analytic at all points inside and on a circle C, having center at $z = a$ and radius r, then at each point z inside C, the function f(z) can be expanded as

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + ... + \frac{f^n(a)}{n!}(z-a)^n + ...$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$$



Example 16: Expand the function $f(x) = \sin x$ about the point $x = \frac{\pi}{6}$.

Soln: $$f(x) = f\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right) f'\left(\frac{\pi}{6}\right) + \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 f''\left(\frac{\pi}{6}\right) + ...$$

$$= \sin\frac{\pi}{6} + \left(x - \frac{\pi}{6}\right)\cos\frac{\pi}{6} + \frac{\left(x-\frac{\pi}{6}\right)^2}{2!}\left(-\sin\frac{\pi}{6}\right) + ...$$

$$= \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2 \frac{1}{2} - \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3 \frac{\sqrt{3}}{2} + ...$$

$$f(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{1}{4\sqrt{3}}\left(x - \frac{\pi}{6}\right)^3 + ...$$

Example 17: Expand the following function $f(z) = \frac{1}{(z-1)(z-4)}$ about z = 2.

Soln: $$\frac{1}{(z-1)(z-4)} = \frac{1}{3}\frac{1}{z-4} - \frac{1}{3}\frac{1}{z-1} = \frac{1}{3}\frac{1}{(z-2)-2} - \frac{1}{3}\frac{1}{(z-2)+1}$$

$$=\frac{1}{3(-2)}\frac{1}{\left[1-\frac{z-2}{2}\right]}-\frac{1}{3}\frac{1}{\left[1+(z-2)\right]}=-\frac{1}{6}\left[1-\frac{z-2}{2}\right]^{-1}-\frac{1}{3}\left[1+(z-2)\right]^{-1}$$

$$=-\frac{1}{6}\left[1+\frac{z-2}{2}+\frac{(z-2)^2}{4}+\frac{(z-2)^3}{8}.......\right]-\frac{1}{3}\left[1-(z-2)+(z-2)^2-(z-2)^3+.....\right]$$

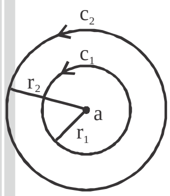
$$=-\frac{1}{2}+\frac{1}{4}(z-2)-\frac{3}{8}(z-2)^2+........$$

Laurent Series expansion:

Laurent's Theorem:

If $f(z)$ is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles C_1 and C_2 with centre at $z=a$ and respective radii r_1 and r_2 $(r_2>r_1)$, then for all z in the region R the Laurent series expansion of $f(z)$ about $z=a$ will be

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

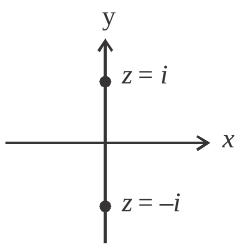


where, $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$

$$b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-a)^{-(n+1)}}dz$$

Note: (i) The Laurrent series converges in the region R.
(ii) The point at which Laurrent series expansion has to be calculated, will be the centre of concentric circles.

Example 18: $f(z)=\frac{1}{1+z^2}$ about $z=i$



Soln: $f(z)=\frac{1}{1+z^2}=\frac{1}{(z+i)(z-i)}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$

$$=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{(z-i)+2i}\right]$$

$$=\frac{1}{2i}(z-i)^{-1}-\frac{1}{2i}\left[\frac{1}{2i\left[1+\frac{(z-i)}{2i}\right]}\right]=\frac{1}{2i}(z-i)^{-1}-\frac{1}{4i^2}\left[1-\frac{(z-i)}{2i}+\frac{(z-i)^2}{4i^2}.....\right]$$

$$=\underbrace{\frac{1}{2i}(z-i)^{-1}}_{\text{Converges for }|z-i|\geq 0}+\underbrace{\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}}_{\text{Converges for }|z-i|<2}$$

So, the function will be convergent for $0\leq|z-i|<2$

Example 19: Exapand the function $f(z)=\frac{1}{(z-1)(z-2)}$ in the annulus $z=1$ and $z=2$.

Soln: $f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$ when $1<|z|<2$, then $\frac{1}{|z|}<1$ & $\frac{|z|}{2}<1$

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+....\right]-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+.....\right]=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example 20: Find the Laurent series of the function $f(z)=\frac{1}{z^2(1-z)}$ about z=0

Soln: $f(z)=\frac{1}{z^2(1-z)}=\frac{1}{z^2}(1-z)^{-1}=\frac{1}{z^2}(1+z+z^2+.....)=\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+........$

Example 21: Expand $\frac{1}{z^2-3z+2}$ for (i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

Soln: $\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$

(i) $0<|z|<1$: $f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)^{-1}}-\frac{1}{-1(1-z)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1}=\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty}z^n$

(ii) $1<|z|<2$: then $\frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$

$\therefore$ $f(z)=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+.....\right)$

(iii) $|z|>2$: then $\frac{2}{|z|}<1$ and $\frac{1}{|z|}<1$

$\therefore$ $f(z)=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+.........\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+..........\right)$

5.5 Singularity of Complex Function

Singular points of a function:

The points at which the function ceases to be analytic, are said to be the singular points of the function.

Example: $f(z)=\dfrac{1}{(z-2)}$ has a singularity at $z=2$.

The singularity will be of the following types:

Isolated singularity:

A point $z=z_0$ is said to be isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at $z=z_0$

(b) $f(z)$ is analytic in the neighbourhood of $z=z_0$ i.e. there exists a neighbourhood of $z=z_0$, containing no other singularity.

Example:

(i) Function $f(z)=\dfrac{1}{z}$ is analytic everywhere except at $z = 0$, therefore $z = 0$ is an isolated singularity.

(ii) The function $f(z)=\dfrac{z+2}{(z-1)(z-2)(z-3)}$ has three isolated singularities at $z = 1, 2$ and 3.

Non-isolated singularity:

A point $z=z_0$ is said to be non-isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at $z=z_0$

(b) there exists a neighbourhood of $z=z_0$, containing other singularities of $f(z)$.

Example: $f(z) = \dfrac{1}{\left[\sin\dfrac{\pi}{z}\right]}$

Condition of singularity: $\sin\dfrac{\pi}{z}=0 \Rightarrow \dfrac{\pi}{z}=n\pi \Rightarrow z=\dfrac{1}{n}\,(n=0,1,2,.....\infty)$

The point $z = 0$ corresponding to $n\to\infty$, will be surrounded by infinite many singular points of $f(z)$. Thus, $z = 0$ is an non-isolated singularity of $f(z)$.

Types of isolated singularity

If $f(z)$ is an isolated singular point at $z = a$, then we can expand $f(z)$ about $z = a$ into Laurent series as:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{\substack{\text{principal part}\\ \text{of the expansion}}}=\left[a_0+a_1(z-a)+a_2(z-a)^2+.....\right]+\left[\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+.....\right]$$

Therefore, three types of singulatiry are as follows:

(1) Removable singularity:

If the principal part of the Laurent series expansion of *f (z)* about z = a contains no term i.e. $b_n = 0$ for all 'n', then *f (z)* has a removable singularity at z = a.

In this case, Laurent series expansion is $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

Example: $f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ....\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - .....$

Since, there is no negative term in the laurent series expansion of *f (z)* about z = 0, hence z = 0 is a removable singularity of *f (z)*.

(2) Non-essential singularity or Pole:

If the principal part of the Laurent series expansion of *f (z)* about z = a contains finite number of terms, say *m*, i.e. $b_n = 0$ for all $n > m$, then *f (z)* has a non-essential singularity or a pole of order *m* at z = a . A pole of order one is also known as simple pole.
Thus if z = a is pole of order m of function *f (z)*, then *f (z)* will have the Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} b_n (z-z_0)^{-n}$$

Example: $f(z) = \frac{z}{(z-1)(z+2)^2}$ has a simple pole at z = 1 and a pole of order 2 at z = -2.

(3) Essential singularity:

If the principal part of the Laurent series expansion of *f (z)* about z = a, contains infinite number of terms i.e. $b_n \neq 0$ for infinitely many values of n, then *f (z)* has an essential singularity at z = a.

Example: $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2\,2!} + ...$

Therefore, *f (z)* has an essential singularity at z = 0.

Example 22: Examine the nature of singularity of the functions: (a) $\sin\left(\frac{1}{1-z}\right)$, (b) $(z-3)\sin\left(\frac{1}{z+2}\right)$.

Soln: (a) $\sin\left(\frac{1}{1-z}\right) = \frac{1}{1-z} - \frac{1}{(1-z)^3 .3!} + \frac{1}{(1-z)^5 .5!} - ....$

so, z = 1 is an isolated essential singular point.

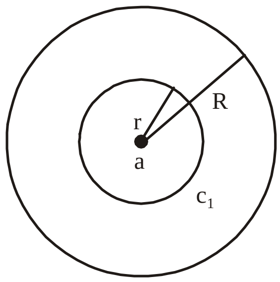
(b) $(z-3)\sin\left(\frac{1}{z+2}\right) = (z-3)\left[\frac{1}{z+2} - \frac{1}{(z+2)^3 .3!} + \frac{1}{(z+2)^5 .5!} - ....\right]$

so, z = -2 is an isolated essential singular point.

5.6 Residue of a Complex Function

Residue at a pole:

Let, z = a be a pole of order 'm' of f (z) and C_1 is a circle of radius 'r' with center at z = a which does not contain singularities except z = a, then f(z) is analytic within the annular region r < | z – a | <R can be expanded into Laurrent series within the annulur region as:



$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Co-efficient b_1 is known as residue of f (z) at z = a i.e. Res. f(z = a) = b_1 = $\frac{1}{2\pi i}\oint_{C_1} f(z)dz$

Methods of finding residues:

CASE 1: Residue at simple pole:

(a) Method 1: Res. f(z = a) = $\underset{z\to a}{\text{Lt}}(z-a)f(z)$

(b) Method 2: If f(z) = $\frac{\phi(z)}{\psi(z)}$ where ψ(a) = 0 but φ(a) ≠ 0, then Res. f(z = a) = $\frac{\phi(a)}{\psi'(a)}$

CASE 2: Residue at a pole of order 'n':

(a) Method 1: Res. f (z = a) = $\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$

(b) Method 2: First put z + a = t and expand it into series, then Res. f (z = a) = co-efficient of 1/t.

CASE 3: Residue at z = ∞ :

$$\text{Res.} f(z=\infty) = \underset{z\to\infty}{Lt}\left[-zf(z)\right]$$

Example 23: Find the singular points of the following function and the corresponding residues:

(a) $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z) = \frac{z^2}{z^2+a^2}$ (c) $f(z) = z^2 e^{1/z}$

Soln: (a) $f(z) = \frac{1-2z}{z(z-1)(z-2)} \Rightarrow$ Poles : $z = 0, z = 1, z = 2$

$$\text{Res. } f(z=0) = \underset{z\to 0}{Lt}(z-0)f(z) = \underset{z\to 0}{Lt}\frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}$$

$$\text{Res. } f(z=1) = \underset{z\to 1}{Lt}(z-1)f(z) = \underset{z\to 1}{Lt}\frac{1-2z}{z(z-2)} = 1$$

$$\text{Res. } f(z=2) = \underset{z\to 2}{Lt}(z-2)f(z) = \underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)} = -\frac{3}{2}$$

(b) $f(z) = \dfrac{z^2}{z^2+a^2} \Rightarrow \text{Poles}: z = ia, z = -ia$

$$\text{Res.} f(z=ia) = \left(\frac{z^2}{2z}\right)_{z=ia} = \frac{1}{2}ia; \text{Res.} f(z=-ia) = \left(\frac{z^2}{2z}\right)_{z=-ia} = -\frac{1}{2}ia$$

(c) $f(z) = z^2 e^{1/z} = z^2\left[1 + \dfrac{1}{z} + \dfrac{1}{z^2.2!} + \dfrac{1}{z^3.3!} + ...\right] \Rightarrow \text{Poles}: z = 0$

$$\text{Res. } f(z=0) = \text{Coefficient of } \frac{1}{z} = \frac{1}{3!} = \frac{1}{6}$$

Cauchy's Residue Theorem:

If f (z) in single-valued and analytic in a simple closed curve 'C, except at a finite number of singular points within C, then

$$\oint_C f(z)dz = 2\pi i(\text{sum of the residues at poles within 'C'})$$

Example 24: Evaluate the integral: $\oint_C \dfrac{4-3z}{z(z-1)(z-3)}dz$ where $|z| = \dfrac{3}{2}$

Soln: $f(z) = \dfrac{4-3z}{z(z-1)(z-3)} \Rightarrow \text{Poles}: z = 0, z = 1, z = 3$

But, the given contour is circle centered at the origin and radius 3/2 units.

Therefore, only z = 0 and z = 1 within the contour.

$$I = 2\pi i[\text{Res.} f(z=0) + \text{Res.} f(z=1)] = 2\pi i\left[\frac{4}{3} - \frac{1}{2}\right] = \frac{5\pi i}{3}$$

Example 25: Evaluate the integral: $\oint_C \dfrac{e^{2z}+z^2}{(z-1)^5}dz$ *where* $|z| = 2$

Soln: $f(z) = \dfrac{e^{2z}+z^2}{(z-1)^5} \Rightarrow \text{Poles}: z = 1\,(\textit{order } 5)$

$$I = 2\pi i \times \text{Res.} f(z=1) = 2\pi i \times \frac{1}{4!}\frac{d^4}{dz^4}\left[e^{2z}+z^2\right]_{z=1} = 2\pi i \times \frac{2e^2}{3} = \frac{4\pi i e^2}{3}$$

Example 26: The value of the integral $\oint_C \dfrac{\text{Sin } z}{z^6}dz$, where C is the circle with centre $z = 0$ and radius 1 unit **[TIFR 2016]**

(a) $i\pi$ (b) $\dfrac{i\pi}{120}$ (c) $\dfrac{i\pi}{60}$ (d) $-\dfrac{i\pi}{6}$

Soln: $\frac{\sin z}{z^6} = \frac{1}{z^6}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\dots\right]$

$$= \left[\frac{1}{z^5} - \frac{1}{z^3 3!} + \frac{1}{5!}\frac{1}{z} - \frac{z}{7!} + \dots\dots\right]$$

The residue at the pole $z = 0$ is coefficient of $\frac{1}{z}$ i.e. $\frac{1}{5!}$

Since, the pole at $z = 0$ lies with in C.

$$\therefore \quad \int_C \frac{\sin z}{z^6} dz = 2\pi i(\text{sum of residues}) = \frac{\pi i}{60}$$

Correct option is (c)

Example 27: The value of integral , $I = \oint_c \frac{\sin z}{2z - \pi} dz$

with c a is circle $|z| = 2$, is **[JEST 2014]**

(a) 0 (b) $2\pi i$ (c) πi (d) $-\pi i$

Soln: $\frac{\sin z}{2z - \pi} = \frac{\sin z}{2\left(z - \frac{\pi}{2}\right)} = \frac{\cos\left(z - \frac{\pi}{2}\right)}{2\left(z - \frac{\pi}{2}\right)} = \frac{\cos t}{2t}$ (Let $z - \frac{\pi}{2} = t$)

$$= \frac{1}{2t}\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots\dots\right] = \frac{1}{2}\frac{1}{t} - \frac{t}{2\times 2!} + \frac{t^3}{2\times 4!} - \frac{t^5}{2\times 6!}$$

The residue at $t = 0$ is the coefficient of $\frac{1}{z}$ i.e. $\frac{1}{2}$

Therefore, the pole $z = \frac{\pi}{2}$ lies within the contour c and the residue at $z = \frac{\pi}{2}$ is $\frac{1}{2}$

$$\therefore \quad \oint_C \frac{\sin z}{2z - \pi} dz = 2\pi i[\text{sum of residue}] = 2\pi i \times \frac{1}{2} = \pi i$$

Correct option is (c)

Definite integrals of trigonometric functions of $\cos\theta$ and $\sin\theta$:

Method:

(i) Consider the contour to be a circle centered at the origin and having radius one unit i.e. |z|=1

(ii) Assume, $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$

(iii) Therefore, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$.

Replacements regarding $\cos\theta$ and $\sin\theta$ is to be done only in the denominator of the given integral.

(iv) The limit will be changed from $0 \to 2\pi$ to $\oint_C$

(v) Find the singular points and find residues at only those singular points which lies inside the unit circle.

(vi) Finally use the Cauchy residue theorem.

Example 28: Evaluate the integral: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$; $a > b > 0$

Soln: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_0^{2\pi} \frac{2dz}{i(bz^2+2az+b)}$

The singular points are at $\Rightarrow z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ & $z = \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

The singular point $z = \beta$ will lie outside the unit circle as a>b>0 while the singular point $z = \alpha$ will lie inside the unit circle which is a simple pole.

$$\text{Res.} f(z=\alpha) = \lim_{z\to\alpha}(z-\alpha) f(z) = \lim_{z\to\alpha} \frac{2}{ib}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{2}{ib(\alpha-\beta)} = \frac{2}{ib}\times\frac{b}{2\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}$$

Therefore, by cauchy Residue theorem.

$$I = 2\pi i \times \text{Residue} = 2\pi i \times \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 29: Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

Soln: $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \oint_C \frac{1}{5+4.\frac{1}{2}\left(z+\frac{1}{2}\right)}.\frac{dz}{iz} = \frac{1}{i}\oint_C \frac{dz}{\left(5z+2z^2+2\right)} = \oint_C \frac{dz}{(2z+1)(z+2)}$

$f(z) = \frac{1}{(2z+1)(z+2)}$ $\Rightarrow$ poles are at $z = -\frac{1}{2}, -2$ (Only z = –½ is within the circle)

Res. f(z = –½) = $\left(z+\frac{1}{2}\right).\frac{1}{(2z+1)(z+2)}\Big|_{z=-\frac{1}{2}} = \frac{1}{2(z+2)}\Big|_{z=-\frac{1}{2}} = \frac{1}{3}$

Hence, the integral = $\frac{2\pi i}{3}$

Example 30: Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{1-2m\cos\theta+m^2}, \left(m^2<1\right)$ **[TIFR 2015]**

Soln: $\int_0^{2\pi} \frac{d\theta}{1-2m\cos\theta+m^2}, \left(m^2<1\right) = \oint_C \frac{dz/iz}{1-2m\times\frac{1}{2}\left[z+\frac{1}{z}\right]+m^2} = \oint_C \frac{dz}{iz\left(z-m\left(\frac{z^2+1}{z}\right)+m^2 z\right)}$

$$= \oint_C \frac{dz}{i\left(z-mz^2-m+m^2 z\right)} = \oint_C \frac{1}{i}\frac{dz}{(z-m)-mz(z-m)} = \frac{1}{i}\oint_C \frac{dz}{(z-m)(1-mz)}$$

$f(z) = \frac{1}{(z-m)(1-mz)}$ has poles of order one at z = m, $\frac{1}{m}$

Since, m < 1 so z = m will be within the circle $|z| = 1$

So, $I = \frac{1}{i}\left[2\pi i \times res.f(z=m)\right] = \frac{1}{i}\left[2\pi i \times \lim_{z\to m}(z-m)\frac{1}{(z-m)(1-mz)}\right]$

$$= \frac{1}{i}\times 2\pi i \times \frac{1}{1-m^2} = \frac{2\pi}{1-m^2}$$

Example 31: Evaluate the integral $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta$

Soln: $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\,d\theta$ = Real part of $\int_0^{2\pi} e^{\cos\theta}e^{i(\sin\theta-n\theta)}d\theta$

= Real part of $\int_0^{2\pi} e^{(\cos\theta+i\sin\theta)}e^{-in\theta}d\theta$ = Real part of $\oint_C e^z . z^{-n}\frac{dz}{iz}$

= Real part of $\frac{1}{i}\oint_C \frac{e^z}{z^{n+1}}dz$

$f(z) = \frac{e^z}{z^{n+1}}$ $\Rightarrow$ poles are at z = 0 of order n $\Rightarrow$ Res. $f(z=0) = \frac{1}{n!}.\frac{d^n}{dz^n}\left(z^{n+1}.\frac{e^z}{z^{n+1}}\right)\Big|_{z=0} = \frac{1}{n!}$

Hence the given integral = real part of $\left(\frac{1}{i}.2\pi i.\frac{1}{n!}\right) = \frac{2\pi}{n!}$

Evaluation of improper integrals between the limit $-\infty$ to $+\infty$:

Theorem 1:

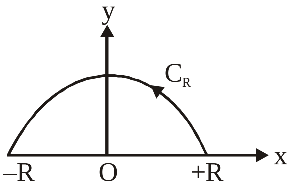
If AB is an arc $(\alpha \le \theta \le \beta)$ of a circle $|z| = R$ and $\lim_{z\to\infty} z\, f(z) = k$(constant). Then,

$$\lim_{R\to\infty}\int_{AB} f(z)\,dz = i(\beta-\alpha)k$$

Theorem 2:

If f(z) is a function of complex variable z which satisfies the following conditions:
(i) f(z) is analytic in the upper half plane except at a finite number of poles.
(ii) f(z) has no poles on the real axis.
(iii) $z\ f(z) \to 0$ uniformly as $|z| \to \infty$

So that $\lim\limits_{R\to\infty} \int_{C_R} f(z)\,dz \to 0$

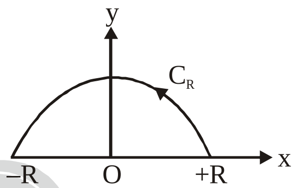


Theorem 3:

Jordan's Lemma: If f(z) is a function of complex variable z which satisfies the following conditions.
(i) f(z) is analytic in the upper half plane except at a finite number of poles.
(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \le \arg z \le \pi$, then

$$\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\,dz \to 0$$

where m is a +ve number and C_R is the semicircle of radius R.



Theorem 4:

If AB is an arc $(\theta_1 \le \theta \le \theta_2)$ of a circle $|z-a| = r$ and $\underset{z\to a}{Lt}(z-a)f(z) = k$, then

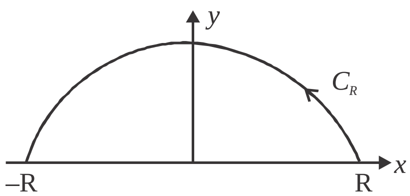
$$\lim_{r\to 0} \int_{AB} f(z)\,dz = i(\theta_2 - \theta_1)k$$

Short Method
Case 1:

If $f(x)$ contains only polynomial terms, then we will take $f(z)$ same as given as $f(x)$. Then find the singular points of $f(z)$ and check which points lie in the upper half plane.

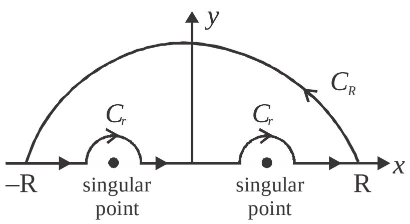
(A) If the singular points does not lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left[\Sigma \text{ Res. at all poles within C}\right] - \left[\lim_{z\to\infty} z\ f(z)\right] i\pi$$



(B) If the singular point lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \pi i\left[\Sigma \text{ Res. at all poles within C}\right] - \left[\lim_{z\to\infty} z\ f(z)\right] i\pi$$



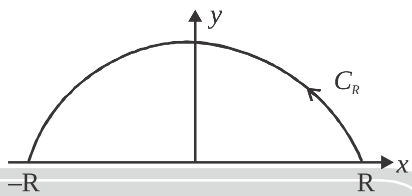
Case 2:

If $f(x)$ contains cosines and sine functions along with polynomial functions then $f(x)$ can be treated as a real or imaginary part of $f(z)$. Then find the singular points of $f(z)$ and check which points lie in the upper half plane.

(A) If the singular points does not lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i\left[\Sigma\text{Res. at all poles within C}\right] \text{ if as } z\to\infty,\ \frac{f(z)}{e^{imz}}\to 0 \text{ and}$$

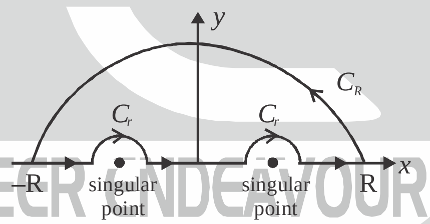
$$\int_{-\infty}^{\infty} f(x)\,dx = \text{Real part or imaginary part of } \int_{-\infty}^{\infty} f(z)\,dz$$



(B) If the singular points lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i\left[\Sigma\text{ Res. at all poles within C}\right] \text{ if as } z\to\infty,\ \frac{f(z)}{e^{imz}}\to 0 \text{ and}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \text{Real part or imaginary part of } \int_{-\infty}^{\infty} f(z)\,dz$$



Example 32: Calculate the following integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Soln: $f(x) = \frac{1}{1+x^2}$

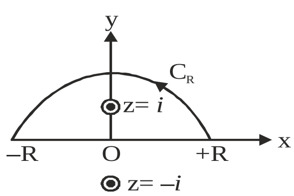
$f(z) = \frac{1}{1+z^2}$ has poles at $z = \pm i$ (Poles are not on real axis)

In this case we will take a contour 'C' consisting of

(i) A semicircle $C_R : |z| = R$ in the upper half of complex plane.

(ii) Real axis from –R to +R

$\oint f(z)\,dz = 2\pi i$ [sum of residues at the poles]



$$\Rightarrow \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i(\text{residue at } z = +i) \quad \text{... (i)}$$

Residue of $f(z)$ at $z=+i=\lim_{z\to +i}(z-i)\dfrac{1}{1+z^2}=\dfrac{1}{2i}$

Taking the limit $R\to\infty$ in the equation (i), we get

$$\Rightarrow \lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}dz+\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=2\pi i\left(\frac{1}{2i}\right)=\pi \qquad \text{... (ii)}$$

C_R is an arc $0\le\theta\le\pi$ of a circle $|z|=R$ and $\lim_{z\to\infty} z\ f(z)=\lim_{z\to\infty}\dfrac{z}{1+z^2}=\lim_{z\to\infty}\dfrac{1/z}{1/z^2+1}=0$

Thus, $\lim_{R\to\infty}\int_{C_R} f(z)dz=i(\pi-0)(0)=0$

Therefore, from equation (ii) $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\pi$

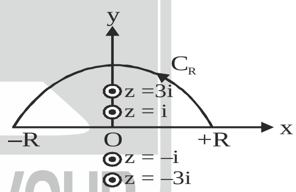
Short Method:

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\int_{-\infty}^{\infty}\frac{dz}{1+z^2}=2\pi i\left[\text{Res. } f(z=i)\right]-\lim_{z\to\infty}\left[zf(z)\right]i\pi=2\pi i\left[\frac{1}{2i}\right]=\pi$$

Example 33: Apply calculus of residues to show that $\displaystyle\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{\pi}{24}$

Soln: $I=\displaystyle\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+9)}$

Let $f(z)=\dfrac{1}{(z^2+1)(z^2+9)}$



The singular points of f(z) are

$(z^2+1)(z^2+9)=0 \quad\Rightarrow\quad z=i,-i,3i,-3i$

The only poles lying within the contour (upper half plane) are at $z=+i$ and +3i

$$\int_{-\infty}^{\infty}f(z)dz=\int_{-\infty}^{\infty}f(x)dx=2\pi i\left(\text{Res. at }(z=i)+\text{Res. at }(z=3i)\right)-\lim_{z\to\infty}\left[zf(z)\right]i\pi$$

Here, $\displaystyle\lim_{z\to 0}zf(z)=\lim_{z\to 0}\frac{z}{(z^2+1)(z^2+9)}=\lim_{z\to 0}\frac{z}{z^4\left(1+\frac{1}{z^2}\right)\left(1+\frac{9}{z^2}\right)}=0$

Therefore, $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=2\pi i\left[\frac{1}{16i}-\frac{1}{48i}\right]=\frac{\pi}{12}\Rightarrow\int_0^{\infty}\frac{dx}{(x^2+1)(x^2+9)}=\frac{\pi}{24}$

Example 34: Prove $\int_{-\infty}^{\infty} \frac{\cos x dx}{\left(x^2+a^2\right)} = \frac{\pi}{a} e^{-a} \quad ; a > 0$

Soln: Let $f(z) = \frac{e^{iz}}{\left(z^2+a^2\right)}$; For singuarities, $z^2 + a^2 = 0 \Rightarrow z = \pm ia$.

The only pole which lies in the upper half of the circle is at z = ia.

Therefore, Residue at z = ia= $\lim_{z \to ia} \frac{(z-ia)e^{iz}}{(z-ia)(z+ai)} = \frac{e^{-a}}{2ia}$

Now by cauchy residue theorem.

$$\int_{C_R} f(z)dz + \int_{-R}^{+R} f(x)dx = 2\pi i(\text{sum of the residues at the poles}) \quad \dots (1)$$

Now in the limit of $R \to \infty$, equation (2) becomes

$$\underset{R\to\infty}{Lim} \int_{C_R} f(z)dz + \int_{-\infty}^{+\infty} f(x)dx = 2\pi i(\text{sum of the residues at the poles}) \quad \dots (2)$$

As $z \to \infty$, $\frac{1}{z^2+a^2} \to \infty$

Therefore, By Jordan's lemma $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{\left(z^2+a^2\right)} dz \to 0$

Therefore, equation (2) gives

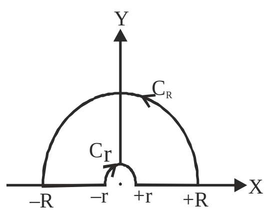
$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \times \frac{e^{-a}}{2ia} \Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{x^2+a^2} = \frac{\pi}{a} e^{-a} \Rightarrow \int_{-\infty}^{\infty} \frac{(\cos x + i \sin x)}{\left(x^2+a^2\right)} = \frac{\pi}{a} e^{-a}$

$\Rightarrow \quad \int_{0}^{\infty} \frac{\cos x}{\left(x^2+a^2\right)} dx = \frac{\pi}{2a} e^{-a}$

Evaluation of some improper integrals in which the pole lies on the real axis:

Example 35: Prove $\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}; \; m > 0$

Soln: Let $f(z) = \frac{e^{imz}}{z}$. The singularity is at z=0 which is at real axis. No pole will be inside the semicircle. Now taking the integral counter clockwise and using the cauchy integral theorem, we get



According to Cauchy Residue theorem, we get

$$\int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz + \int_{r}^{R} f(x)dx = 0 \; (\text{Since there is no pole inside the contour})$$

Taking the limit $R \to \infty$ and $r \to 0$, we get