Chapter 1

Mathematical Logic

Proposition & Predicate Calculus:

Proposition is a declarative sentence that can be true or false but not both.

Eg. (a) 1 + 1 = 2 True - Proposition

(b) 1 + 2 = 5 False - Proposition

(c) Washington is a capital of USA - Proposition

(d) What time it is? (question) - Not Proposition

Propositions can be combined with logical operators.

1. **Negative ($\neg$P)** : Not P

It is not the case that P

P	$\neg P$
$O(F)$	T
T	$O(F)$

Truth table

Example:

P : Today is Friday

$\neg$P : It is not Friday today

2. **Conjunction** $(P \wedge Q)$: P and q

P but q

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth table

Example:

P : Today is Friday

q : It is raining

$P \wedge q$: Today is Friday and it is raining.

3. **Disjunction** $(P \vee q)$ P or q

(Inclusive OR)

P	q	$P \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Truth table

Example:

P : Student with Mathematics can take this class

q : Students with English can take this class

P or q : Student with Mathematics or English can take this class.

4. **Exclusive OR** $(\oplus)$: P or q but not both

P	q	$P \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Truth table

5. **Implication or Conditional** $(\rightarrow)$: $P \rightarrow Q$

$\rightarrow$ If P then q

$\rightarrow$ q if P

$\rightarrow$ q when P

$\rightarrow$ q provided P

P	q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

Truth table

Example:

P : It is raining

q : There will be flood

$P \rightarrow q$: If it is raining then there will be flood

6. **Bi-conditional** $(p \leftrightarrow q)$ can be also written as, $(p \rightarrow q) \wedge (q \rightarrow p)$

$\rightarrow$ p is equivalent to q

$\rightarrow$ p if and only if q

$\rightarrow$ p is necessary and sufficient for q

$\rightarrow$ p iff q

P	q	$P \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Truth table

Converse, Contrapositive and Inverse:

$P \to q$ Implication

$q \to P$ Converse

$\neg q \to \neg P$ s Contrapositive

$\neg P \to \neg q$ Inverse

Also,

Implication $\equiv$ Contrapositive (Logically equivalent)
Converse $\equiv$ Inverse

Example: Find the converse, contrapositive and inverse of: If it is raining then the home team wins.

Soln. P : It is raining

q : The home team wins

Contrapositive $(\neg q \to \neg p)$: If the home team does not wins then it is not raining

Converse $(q \to p)$: If the home team wins it is raining.

Inverse $(\neg p \to \neg q)$: If it is not raining, then the home team does not wins.

Well formed formulas (Wff).

1. Each atomic formula is wff
2. If A and B are wff then $\neg A$, $A \cup B$, $A \leftrightarrow B$, $A \to B$, $A \cap B$
3. All wff by repeating steps (1) and (2).

Tautology and Contradiction: A proposition function whose truth value is true under all possible value of the proposition variable is called tautology.

Equation: $P \vee \neg P$

$\to$ The proposition function with truth value false for all possible values of proposition variables is called contradiction.

Eg. $P \wedge \neg P$

$\to$ Otherwise it is called contingency.

Duality Law: Two proposition function are said to be dual of each other if one can be obtained from another by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F and F by T.

Eg. $P \wedge Q$ (dual) $P \vee Q$

Some Equivalent Formula's:

			Dual
1.	**Commutative Law:**	$P \vee Q = Q \vee P$	$P \wedge Q = Q \wedge P$
2.	**Associative Law:**	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
3.	**Distributive Law:**	$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

4. **Demorgan's Law:** $\neg(P \vee Q) = \neg P \wedge \neg Q$ $\neg(P \vee Q) = \neg P \wedge \neg Q$

5. **Double Negation:** $\neg(\neg P) = P$

6. **Implication Law:** $P \to Q = \neg P \vee Q$

7. **Contrapositive:** $P \to Q = \neg Q \to \neg P$

8. **Absorption Law:** $P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) = P$

Propositional Logic:

Valid Argument: An argument is a sequence of statements. It is said to be valid iff all the premises are true implies that conclusion must be true.

Validity can be proven using rules of inference.

Rules of Inference:

1. **Simplification:** $\dfrac{P \wedge q}{\therefore P}$ or $\dfrac{P \wedge q}{\therefore q}$

2. **Implication :** $\dfrac{\neg P}{\therefore P \to q}$ or $\dfrac{q}{\therefore P \to q}$

3. **Modus Pones :** $\begin{array}{c} P \\ P \to q \\ \hline \therefore q \end{array}$

4. **Modus Tollens :** $\begin{array}{c} P \to q \\ \neg q \\ \hline \therefore \neg P \end{array}$

5. **Hypothetical Syllogism:** $\begin{array}{c} P \to q \\ q \to r \\ \hline \therefore P \to r \end{array}$

Predicate Calculus / First Order Logic:

A part of declarative sentence describing the properties of an object / relationship among objects is called predicate.

Eg. Ram is a bachelor

$P(x): x$ is a bachelor

$\Rightarrow$ where x is a predicate variable

Quantifier: One way to convert predicate into proposition is using qualifier.

Two main qualifiers:

(a) **Universal qualifier ($\forall$) used for:**

- for all x - for any x
- for every x - for arbitrary x
- for each x

Eg. x is greater than 3 be a predicate

$P(x): x > 3$. What is truth value of $\forall x P(x)$. Domain of x - Natural number

Soln. $\forall x P(x)$ - False. Counter example-2.

(b) **Existential Qualifier** $(\exists)$:

– There exist an x

– For some x

– There is an x

Eg. In above example:

$\exists x\, P(x)\, st. P(x) : x > 3$

is true for Eg. for $x = 4$

Other Quantifier:

→ **Uniqueness quantifier** $(\exists! x)$: Exactly one-one and only one x

$\exists! x[x = 3]$ true

→ It can be expressed in terms of $\forall$ and $\exists$

$\exists! x\; p(x)$ can be written as,

$\exists x[P(x) \wedge \forall y\{p(y) \Rightarrow x = y\}]$

Negation of Qualifier: Suppose we have a statement, all students in the class have taken DM.

It can be written as,

$\forall x P(x) - x$ has taken DM.

Its negation can be stated as,

→ It is not the case that all students in the class has taken a course in DM

or

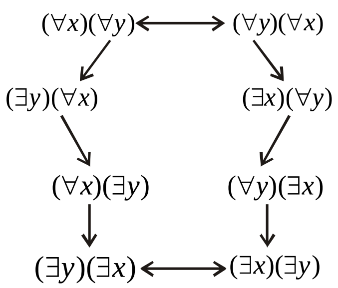
→ There is some student in the class who has not taken DM.

$\therefore \exists x \neg P(x)$

Thus ,

$$\boxed{\begin{array}{l}\neg \forall x P(x) = \exists x \neg P(x) \\ \neg \exists x P(x) = \forall x \neg P(x)\end{array}}$$

Multiple Qualifier: The relationship between multiple qualifier can be stated as.



SOLVED EXAMPLES

1. What is contrapositive of following sentence?
If I am feeling well then you come.

Soln. P : If I am feeling well
Q: You come
Contrapositive: $\neg q \to \neg P$
If you don't come then I am not feeling well.

2. What is truth value of $\neg(p \vee q) \vee [(\neg p) \wedge q] \vee p$

Soln. $= \neg(p \vee q) \vee [(\neg p \vee p) \wedge (p \vee q)]$
$= \neg(p \vee q) \vee [T \wedge (p \vee q)]$
$= \neg p \vee q) \vee (p \vee q)$
$\Rightarrow$ If $p \vee q$ is taken as X
$= \neg XVX$
$= T \Rightarrow$ Tautology

3. Find the conclusion using the following premises - $\neg p \wedge q,\ r \to p,\ \neg r \to s,\ s \to t$

Soln.

1.	$\neg p \wedge q$	Premise
2.	$\neg p$	Simplification on 1
3.	$r \to p$	Premise
4.	$\neg p$	Modus tolens on 2, 3
5.	$\neg r \to s$	Premise
6.	s	Modus pones on 4, 5
7.	$s \to t$	Premise
8.	t	Modus pones on 7, 6

4. Is the formula $p \to (p \vee q)$ valid

Soln. $p \to (p \vee q)$
$\Rightarrow \neg p \vee (p \vee q)$ $\quad (p \to q = \neg p \vee q)$
$\Rightarrow (\neg p \vee p \vee q)$
$\Rightarrow (T \vee q)$
$\Rightarrow$ Tautology so, valid

5. Check whether the following conclusion valid or not
'If you send me email then I will finish writing the program. If you don't send me email I will go to bed early. If I go to bed early then I will feel refreshed'.
Conclusion: If I don't finish writing the program then I will wake up feeling refreshed.

Soln. P : You send me email
q : I will finish writing program
r : I will go to bed early
s : I will wake up feeling refreshed
Premise: $p \to q,\ \neg p \to r,\ r \to s$
Conclusion: $\neg q \to s$

1. $p \to q$ Premise
2. $\neg q \to \neg p$ Contrapositive on (1)
3. $\neg p \to r$ Premise
4. $\neg q \to r$ Hypothetical syllogism on (2) & (3)
5. $r \to s$ Premise
6. $\neg q \to s$ Hypothetical syllogism on (4) & (5)

Hence, valid.

6. Find the truth value of

(a) $\exists y \forall x [x + y = 0]$

(b) $\forall x \exists y [x + y = 0]$

Soln. (a) $\exists y \forall x [x + y = 0]$ False

Value of y can not be chosen independently of x

(b) $\forall x \exists y [x + y = 0]$ True

$y = -x$

7. Find the truth value of

(a) $\forall y \exists! x [x + y < 0]$

(b) $\exists! x \forall y [x \cdot y = 0]$

(c) $\forall y \exists! x [x \cdot y = 0]$

Soln. (a) $\forall y \exists! x [x + y < 0]$ False

$y = 2$ x can be –3, –4, –5

(b) $\exists! x \forall y [x \cdot y = 0]$ True

for $x = 0$

(c) $\forall y \exists! x [x \cdot y = 0]$ False

If $y = 0$ x can be 0, 1, 2 .....

8. What is the value of $m \forall x [F(x)]$

Soln. $\exists x [\sim F(x)]$