Chapter 7

Linear Programming

To define anything non-trivial like beauty or mathematics is very difficult indeed. Here is a reasonably good definition of operations Research:

Definition-1: Operations Research (OR) is an interdisciplinary branch of applied mathematics and formal science that uses methods like mathematical modeling, statistics, and algorithms to arrive at optimal or near optimal solution to complex problems.

Definition-2: problematic: to grasp it we already have to know, e.g. what is formal science or near optimal. Form a practical point of view. OR can be defined as an art of optimization. i.e. an art finding minima or maxima of some objective function and to some extent an art of defining the objective functions. Typical objective functions are

(i) profit (ii) assembly line performance (iii) crop yield (iv) bandwidth.
(v) loss. (vii) waiting time in queue (viii) risk

From an organizational point of view. OR is something that help management achieve its goals using the scientific process.

The terms OR and Manager Science (MS) are often used synonymously. When a distinction is drawn. Manager science generally implies a closer relationship to Business Management. OR also closely relates to Industrial Engineering. Industrial engineering takes more of an engineering point of view and industrial engineers typically consider OR techniques to be a major part of their tool set. Recently the term Decision Science (DS) has also be coined to OR.

OR Motto and Linear Programming: The most common OR tool is Linear Optimization or Linear Programming (LP).

Remark-1: The "Programming" in Linear Programming is synonym for "optimization". It has at least historically nothing to do with computer programming.

LP is the OR its favourite tool because it is

- Simple
- easy to understand
- robust

"Simple means easy to implement. "Easy to understand means easy to explain (to you boss), and robust" means that it's like the Swiss Army Knife; perfect for nothing, but good enough for everything.

Unfortunately. almost no real- world program is really a linear one thus LP is perfect for nothing . However most real- world problems are 'close enough" to linear one thus LP is good enough for everything Example below elaborates this point.

Example-1: Mr. Quine sells gavagai. He will sell one gavagai for 10 Euros. So, one might expect that buying gavagais from Mr. Quine would cost according to the linear rule 10x Euros.

The linear rule in **Example-1** may well hold for buying 2, 3, or 5 even 50 gavagai But

• One may get a discount if one buys 500 gavagai.
• There are only 1, 000, 000 gavagais in the world. So, the price for 1, 000, 001 gavagais is +x :

• The unit price of gavagais may go up as they became scarce. So, buying 950, 000 gavagai might be considerably more expensive than C9,500, 000.
• It might be pretty hard to buy 0.5 gavagais. Since half a gavagai is no longer a gavagai (gavagai are bought for pets, and not for food).
• Buying –10 gavagai is in principle all right. That would simply mean selling 10 gavagais. However. Mr. Quine would most likely not buy gavagais with the same price he is selling them

OR Motto. It's better to be quantitative and native then qualitative and profound

History of Operation Research(OR)

Prehistory:
Some say that Charles Babbage (1791 -1871) – who is arguably the 'father of computers" is also the "father of operations research" because his research into the cost of transportation and sorting of mail led to England's universal "penny Post" in 1840.

OR During World War II:
This modern field of OR arose during World War II. Scientists in the United Kingdom including Patrick Blackett. Cecil Gordon. C. H. Waddington, Owen Wansbrough-Jones and Frank Yates, and in the United States with George Dantzing looked for ways to make better decisions in such areas as logistics and training schedules. Hence are examples of OR studies done during World War II.

• Britain introduced the convoy system to reduce shipping losses, but while the principle of using warships to accompany merchant ship was generally accepted it was unclear whether it was better for convoys to be small or large. Conveys to be small or large. Convoys travel at the speed of the slowest member, so small convoys can travel faster. It was also argued that small convoys would be harder for German U-boats to detect. On the other hand, large convoys could deploy more warships against an attacker. It turned out in OR analysis that the loses suffered by convoys depended largely on the number of escort vessels present, rather than on the overall size of the convoy. The conclusion, therefore was that a few large convoys are more defensible than many small ones.

• In another OR study a survey carried out by RAF Bomber Command was analyzed. For the survey, Bomber Command inspected all bombers returning from bombing raids over Germany over a particular period. All damage inflicted by German air defenses was noted and the recommendation was given that armor be added in the most heavily damaged areas. OR team instead made the surprising and counter-intuitive recommendation that the armor be placed in the areas which were completely untouched by damage. They reasoned that the survey was biased. since it only included aircraft that successfully came back from Germany. The untouched areas were probably viral areas, which if hit would result in the loss of the aircraft.

• When the Germans organized their air defenses into the Kammhuber Line, it was realized that if the RAF bombers were to fly in a bomber stream they could overwhelm the night fighters who flew in individual cells directed to their targets by ground controllers. It was then a matter of calculating the statistical loss from collisions against the statistical loss from night fighters or calculator how closed the bombers should fly to minimize RAF losses.

Phases of Operations Research Study: Seven step of OR Study:

An OR project can be split in the following seven steps:
Step-1: Formulate the problem: The OR analyst first defines the organization's problem. This includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step-2: Observe the system: Next the OR analyst collected data to estimate the values of the parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and o evaluate (in step 4) a mathematical model of the organization's problem.

Step-3: Formulate a mathematical model of the problem : The OR analyst developed an idealized representation i,e, a mathematical model of the problem.

Step-4: Verify the model and use it for rediction: The OR analystries to determine if the mathematical model developed in Step 3 is an accurate representation of the reality. The verification typically includes observing the system to check if the parameters are correct. It the model does not represent the reality will enough then the OR analyst goes back either to step-3 or Step-2.

Step-5 : Select a suitable alternative : Given a model and a set of alternative. The analyst now chooses the alternative that best meets the organization 's objectives, Sometimes there are many best altematives, in which case the OR analyst should present then all to the organization's decision-makers, or ask for more objectives or restrictions.

Step-6 : Present the result and conclusions : The OR analyst presents the model and recommendations from step 5 to the organization's decision markers. At this point the OR analyst may find that the decision markers do not approve of the recommendations. This may result from incorrect definition of the organization's problem or decision-markers may disagree with the parameters or the mathematical model. The OR analyst goes back to step1. Step 2, step 3. depending on where the disagreement lies.

Step-7 : Implement and evaluate recommendation: Finally, when the organization has accepted the study the OR analyst helps in implementing the recommendations. The system must be constantly monitored and updated dynamically as the environment changes. This means going back to Step 1, Step 2, step3, from time to time.

Linear Programming

Example towards Linear Programming

Very Naive Problem:

Example-2: Tela Inc. manufactures two product # 1 and #2; To manufacture on unit of product # 1 costs C40 and to manucapture one unit of product #2 costs C60. The profit from product # 1 is C30, and the profit from product #2 is C20.

The company wants to maximize its profit how many products # 1 and # 2 should it manufacture?

The solution is trivial; There is no bound on the amount of units the company can manufacture. So it should manufacture infinite number of either product #1 ro #2, or both. If there is a constraint on the number of units manufactured then the company should manufacture only product #1, and not product # 2, This constrained case is still rather trivial.

Less Naive Problem:

Things because more interesting and certainly realistic when there are restrictions in the resources

Example-3: Tela Inc. in **Example-2** can invest C40, 000 in production and use 85 hours of labor. To manufacture one unit of product #1 requires 15 minutes of labor, and to manufacture one unit of product #2 requires 9 minuts of labor.

The company wants to maximize its profit. How many units of product #1 and product # 2 should it manufature? What is the maximuzed profit?

The rather trivial solution of **Example -3** is not applicable now. Indeed, the company does not have enough labour to put all the C40, 000 in product #1.

Since the profit to be maximized depend on the number of 'product # 1 and #2, our decision variables are:

x_1 = number of product # 1 produced; x_2 = number of product # 2 produced.

So the situation is : We want to maximize (max)

profit $30x_1 + 20x_2$

Subject to (s.t.) the constraints

$$\begin{aligned} \text{Money}: &\quad 40x_1 + 60x_2 \le 40,000 \\ \text{labor} &\quad 15x_1 + 9x_2 \le 5,100 \quad (85\times 60 \text{ Min.}) \\ \text{non}-\text{negativity}: &\quad x_1, x_2 \ge 0 \end{aligned}$$

Note the last constraint $x_1, x_2 \ge 0$. The problem does not state this explicitly, but it's implied we are selling products #1 and #2, not buying them.

Remark. Some terminology : The unknowns x_1 and x_2 are called decision variables. The function $30x_1 + 20x_2$ to be maximized is called the objective function

What we have now is a Linear Program (LP). or a Linear Optimization problem.

$$\begin{aligned} \max: &\quad = 30x_1 + 20x_2 \\ \text{s.t.} &\quad 40x_1 + 60x_2 \le 40,000 \\ &\quad 15x_1 + 9x_2 \le 5,100 \\ &\quad x_1, x_2 \ge 0 \end{aligned}$$

We will later see how to solve such LP,s For now we just show the solution. For decision variables it is optimal to produce no product # 1 and thus put all the resource to product #2 which means producing 566.667 number of product #2. The profit will then be C11, 333. 333. In other words the optimal solution is

$x_1 = 0$; $x_2 = 566.667$; $z = 11.333, 333$.

Remark: If it is not possible to manufacture fractional number of produces e.g. 0.667 units then we have to reformulate the LP- problem above to an Integer Program (IP)

$$\begin{aligned} \max: &\quad = 30x_1 + 20x_2 \\ \text{s.t.} &\quad 40x_1 + 60x_2 \le 40,000 \\ &\quad 15x_1 + 9x_2 \le 5,100 \\ &\quad x_1, x_2 \ge 0 \\ &\quad x_1, x_2 \text{ are integers} \end{aligned}$$

We will later know how to solve such IPs (which is more difficult than solving L.P.s) For now we just show the solution:

$x_1 = 1$; $x_2 = 565$; $z = 11, 330$.

FORMULATION OF THE PROBLEM

To begin with, a problem is to be presented in a linear programming form which requires defining the variables involved, establishing relationships between them and formulating the objective function and the constraints. We illustrate this through a few examples, wherein the stress will be on the analysis of the problem and formulation of the linear programming model.

Exp.1. A manufacturer produces two types of models M_1 and M_2. Each M_1 model requires 4 hours of grinding and 2 hours of polishing; whereas each M_2 model requires 2 hours of grinding and 5 hours of polishing The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is Rs. 3 and on an M_2 model is Rs. 4. Whatever is produced in a week is sold in the market How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week.

Soln. Let x_1 be the number of M_1 models and x_2, the number of M_2 models produced per week. Then the weekly profit (in Rs.) is

$$Z = 3x_1 + 4x_2 \quad \text{... (i)}$$

To produce these number of models, the total number of griinding hours needed per week

$$= 4x_1 + 2x_2$$

and the total number of polishing hours required per week

$$= 2x_1 + 5x_2$$

Since the number of grinding hours available is not more than 80 and the number of polishing hours is not more than 180, therefore

$$4x_1 + 2x_2 \le 80 \qquad \text{...(ii)}$$

$$2x_1 + 5x_2 \le 180 \qquad \text{...(iii)}$$

Also since the negative number of models are not produced, obviously we must have

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0 \qquad \text{...(iv)}$$

Hence this allocation problem is to find x_1, x_2 which

maximize $Z = 3x_1 + 4x_2$

subject to $4x_1 + 2x_2 \le 80,\ 2x_1 + 5x_2 \le 180,\ x_1, x_2 \ge 0$.

Objective: The variables that enter into the problem are called decision variables.

The expression (i) showing the relationship between the manufacturer's goal and the decision variables, is called the ***objective function***.

The inequalities (ii), (iii) and (iv) are called the constraints.

The objective function and the constraints all linear, it is a linear programming problem (L.P.P.). This is an example of a real situation from industry.

GRAPHICAL METHOD

Linear programming problems involving only two variables can be effectively solved by a graphical technique. In actual practice, we rarely come across such problems. Even then, the graphical method provides a pictorial representation of the solution and one gets ample insight into the basic concepts used in solving large L.P.P.

Working procedure to solve a linear programmng problem graphically :

Step 1. formulate the givenn problem as a linear programming problem.

Step 2. Plot the given constraints as equalities on x_1x_2-coordinate plane and determine the convex region* formed by them.

Step 3. Determine the vertices of the convex region and find the value of the objective function at each vertex. The vertex which gives the optimal (maximum or minimum) value of the objective function gives the desired optimal solution to the problem.

Otherwise. Draw the dotted line through the origin representing the objective function with Z = 0. As Z is increased from zero, this line moves to the right remaining parallel to itself. We go on sliding this line (parallel to itself), till it is farthest away from the origin and passes through only one vertex of the convex region. This is the vertex where maximum value of Z is attained.

When it is required to minimize Z, value of Z is increased till the dotted line passes through the nearest vertex of the convex region.

Exp.1 Given

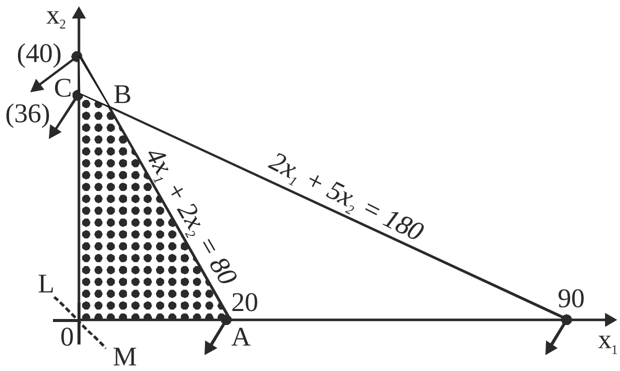
Maximize $Z = 3x_1 + 4x_2$ ...(i)

subject to $4x_1 + 2x_2 \le 80$ ...(ii)

$2x_1 + 5x_2 \le 180$ ...(iii)

$x_1, x_2 \ge 0$ ...(iv)

Soln. The problem is :



Consider x_1x_2-coordinate system as shown in figure above. The non-negativity restrictions (iv) imply that the values of x_1, x_2 lie in the first quadrant only.

We plot the lines $4x_1 + 2x_2 = 80$ and $2x_1 + 5x_2 = 180$.

then any point on or below $4x_1 + 2x_2 = 80$ satisfies (ii) and any point on or below $2x_1 + 5x_2 = 180$ satisfies (iii). This shows that the desired point (x_1, x_2) must be some where in the shaded convex region OABC. This region is called the solution space or region of feasible solutions for the given problem. Its vertices are O(0, 0), A(20, 0) B(2.5, 35) and C(0, 36).

The values of the objective function (i) at these points are

$$Z(O) = 0,\ Z(A) = 60,\ Z(B) = 147.5,\ Z(C) = 144.$$

Thus the maximum value of Z is 147.5 and it occurs at B. Hence the optimal solution to the problem is

$$x_1 = 2.5,\ x_2 = 35 \quad \text{and} \quad Z_{\max} = 147.5.$$

*A region or a set of points is said to be convex if the line joining any two of its points lies completely in the region (or the set). figure(i), figure(ii) represent convex regions while figure(iii) and figure(iv) do not form convex sets.

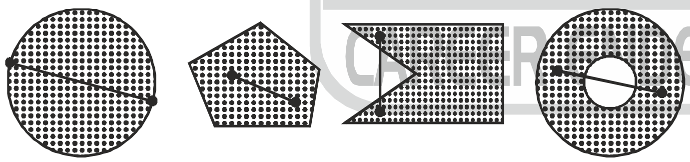


figure (i) figure (ii) figure (iii) figure (iv)

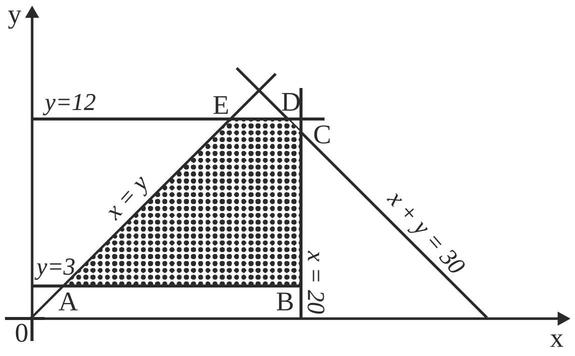
Otherwise. Our aim is to find the point in the solution space which maicmizes the profit function Z. To do this, we observe that on making Z = 0, (i) becomes $3x_1 + 4x_2 = 0$ which is represented by the dotted line LM through O. As the value of Z is increased, the line LM starts moving parallel to itself towards the right. Larger the value of Z, more will be the company's profit. In this way, we go on sliding LM till it is farthest away from the origin and passes through one of the corners of the convex region. This is the point where the maximum value of Z is attained. Just possible, such a line may be one of the edges of the solution space. In that case every point on that edge gives the same maximum value of Z.

Here $Z_{\max}$ is attained at B(2.5, 35). Hence the optimal solution is $x_1 = 2.5,\ x_2 = 35$ and $Z_{\max} = 147.5$.

Exp.2: Find the maximum value of $Z = 2x + 3y$

subject to the constraints : $x + y \le 30,\ y \ge 3,\ 0 \le y \le 12,\ x - y \ge 0,$ and $0 \le x \le 20$.

Soln. Any point (x, y) satisfying the conditions $x \ge 0, y \ge 0$ lies in the first quadrant only. Also since



$x+y\le 30, y\ge 3, y\le 12, x\ge y$ and $x\le 20$, the desired point (*x*, *y*) lies within the convex region ABCDE (shown shaded in above figure). Its vertices are A(3, 3), B(20, 3), C(20, 10), D(18, 12) and E(12, 12).

The values of Z at these five vertices are $Z(A)=15, Z(B)=49, Z(C)=70, Z(D)=72$, and $Z(E)=60$.

Since the maximum valuee of *Z* is 72 which occurs at the vertex *D*, the solution to the L.P.P. is

$$x=18, y=12 \text{ and maximum } Z=72.$$

Exp.3: A company making cold drinks has two bottling plants located at towns T_1 and T_2. Each plant produces three drinks A, B and C and their production capacity per day is shown below :

Cold drinks	*Plant at*	
	T_1	*T_2*
A	6,000	2,000
B	1,000	2,500
C	3,000	3,000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants of T_1 and T_2 are Rs. 6000 and Rs. 4000 respectively. Find (graphically) the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Soln. Let the plants at T_1 and T_2 be run for x_1 and x_2 days. Then the objective is to minimize the operation costs i.e.

$$\min.\ Z=6000x_1+4000x_2 \quad \text{...(i)}$$

Constraints on the demand for the three cold drinks are :

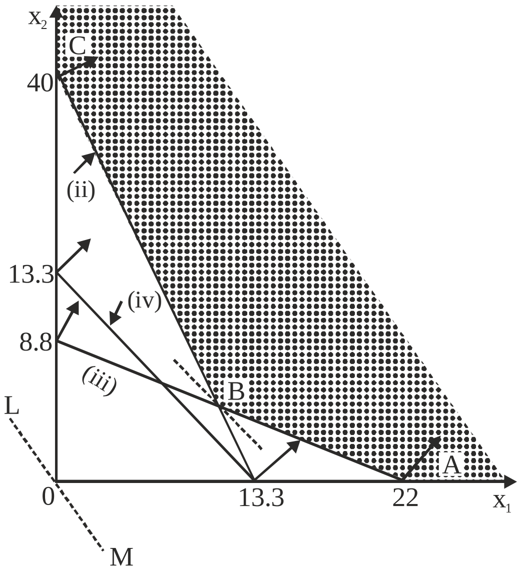
for A, $6,000x_1+2,000x_2\ge 80,000$ or $3x_1+x_2\ge 40$ ...(ii)

for B, $1,000x_1+2,500x_2\ge 22,000$ or $x_1+2.5x_2\ge 22$ ...(iii)

for C, $3,000x_1+3,000x_2\ge 40,000$ or $x_1+x_2\ge 40/3$ ...(iv)

Also $x_1, x_2\ge 0$. ...(v)

Thus the L.P.P. is to minimize (i) subject to constraints (ii) to (v).



The solution space satisfying the constraints (ii) to (v) is shown shaded in in figure above. As seen from the direction of the arrows, the solution space is unbounded. The constraint (iv) is dominated by the constraints (ii) and (iii) and hence does not affect the solution space. Such a constraint as (iv) is called the redundant constraint.

The vertices of the convex region ABC are A(22, 0), B(12, 4) and C(0, 40).

Values of the objective function (i) at these vertices are

$Z(A) = 132,000, Z(B) = 88,000, Z(C) = 160,000$.

Thus the minimum value of Z is Rs. 88,000 and it occurs at B. Hence the solution to the problem is

$x_1 = 12$ days, $x_2 = 4$ days, Z_{min} = Rs. 88,000.

Otherwise. Making Z = 0, (i) becomes $3x_1 + 2x_2 = 0$ which is represented by the dotted line LM through O. As Z is increased, the line LM moves parallel to itself, to the right. Since we are interested in finding the minimum value of Z, value of Z is increased till LM passes through the vertex nearest to the origin of the shaded region i.e. B(12, 4).

Thus the operating cost will be minimum for $x_1 = 12$ days, $x_2 = 4$ days and $Z_{min} = 6000 \times 12 + 4000 \times 4 =$ Rs. 88,000.

Obs. The dotted line parallel to the line LM is called the iso-cost line since it represents all possible comb inations of x_1, x_2 which produce the same total cost.

SOME EXCEPTIONAL CASES

The constraints generally, give region of feasible solution which may be bounded or unbounded. In problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. In fact, this is true for all L.P. problems for which solutions exist. Thus it may be stated that if there exists an optimal solution of an L.P.P., it will be at one of the vertices of the solution space.

In each of the above examples, the optimal solution was unique. But it is not always so. In fact, L.P.P. may have

(i) a unique otimal solution,

or (ii) an infinite number of optimal solutions (also called, alternate solutions)

or (iii) an unbounded solution,

or (iv) no solution.

We now give below a few examples to illustrate the exceptional cases (ii) to (iv).

Exp.1. A firm uses milling machines, grinding machines and lathes to produce two motor parts. The machining times required for each part, the machining times available on different machines and the profit on each motor part are given below :

Type of machine	Machining time required for the motor part (mts)		Maximum time available per week (minutes)
	I	II	
Milling machines	10	4	2,000
Grinding machines	3	2	900
Lathes	6	12	3,000
Profit / unit (Rs.)	100	40	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

Soln. Let x_1, x_2 be the number of parts I and II manufactured per week. Then objective being to maximize the profit, we have

maximize $Z = 100x_1 + 40x_2$ ...(i)

Constraints being on the time available on each machine, we obtain

for milling machines, $10x_1 + 4x_2 \le 2,000$ ...(ii)

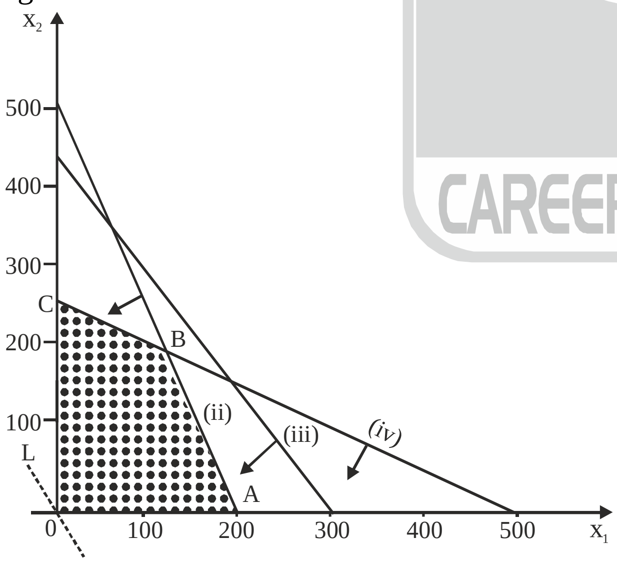
for grinding machines, $3x_1 + 2x_2 \le 900$ ...(iii)

for lathes, $6x_1 + 12x_2 \le 3,000$ ...(iv)

Also $x_1, x_2 \ge 0$ ...(v)

Thus the problem is to determine x_1, x_2 which maximize (i) subject to the constraints (ii) to (v).

The solution space satisfying (ii), (iii), (iv) and meeting the non-negativity restrictions (v) is shown shaded in figure below.



Note that (iii) is a redundant constraint as it does not affect the solution space. The vertices of the convex region OABC are

O(0, 0), A(200, 0), B(125, 187.5), C(0, 250).

Values of the objective function (i) at these vertices are

$Z(O) = 0, Z(A) = 20,000, Z(B) = 20,000$ and $Z(C) = 10,000$

Thus the maximum value of Z occurs at two vertices A and B.

$\therefore$ Any point on the line joining A and B will also give the same maximum value of Z i.e. there are infinite number of feasible solutions which yield the same maximum value of Z.

Thus there is no unique optimal solution to the problem and any point on the line AB can be taken to give the profit of Rs. 20,000.

Obs. An L.P.P. having more than one optimal solution, is said to have alternative or multiple optimal slutions. It implies that the resources can be combined in more than one way to maximize the profit.

Exp.2 Using graphical method, solve the following L.P.P. :

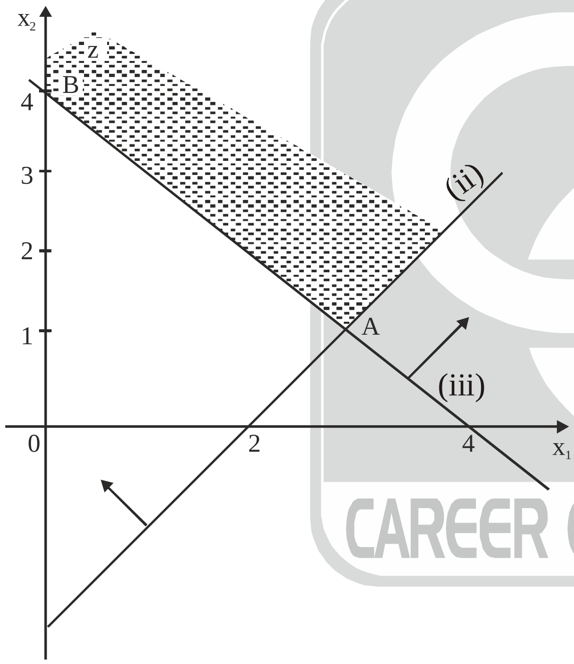
maximize $z = 2x_1 + 3x_2$ (i)

subject to $x_1 - x_2 \le 2$ (ii)

$x_1 + x_2 \ge 4$ (iii)

$x_1, x_2 \ge 0$ (iv)

Soln. Consider x_1x_2-coordinate system. Any point (x_1, x_2) satisfying the restrictions (iv) lies in the first quadrant only. the solution space satisfying the constraints (ii) and (iii) is the convex region shown shaded in figure below.



Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are A(3, 1) and B(0, 4).

Values of the objective function (i) at these vertices are Z(A) = 9 and Z(B) = 12.

But there are points in this convex region for which Z will have much higher values. For instance, the point (5, 5) lies in the shaded region and the value of Z thereat is 12.5. In fact, the maximum value of Z occurs at infinity. Thus the problem has an unbounded solution.

Exp. 3. Max $z = x_1 + x_2$ ... (i)

S.t. $x_1 - x_2 \ge 0$ ... (ii)

$-3x_1 + x_2 \ge 3$ ... (iii)

$x_1, x_2 \ge 0$ ... (iv)

Soln. Using the graphical method,