Region of convergence $|z|<1$

This series is convergent within a circle of radius 1 and centre (0, 0).

Example 32: Find the radius of convergence of the series

$$\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+......$$

Soln: The coefficient of z^n of the given power series is given by

$$a_n=\frac{1.3.5...(2n-1)}{2.5.8...(3n-1)}$$

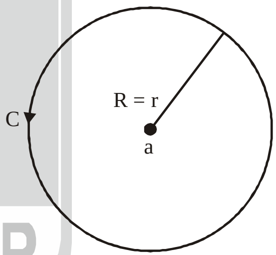
$$a_{n+1}=\frac{1.3.5...(2n-1)(2n+1)}{25.8...(3n-1)(3n+2)}$$

So, $$\frac{a_{n+1}}{a_n}=\frac{2n+1}{3n+2}=\frac{2}{3}.\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}\Rightarrow R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{3}{2}\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}\right|=\frac{3}{2}$$

■ Taylor Series Expansion

If a function $f(z)$ is analytic at all points inside the circle C, having center at $z=a$ and radius r, then at each point z inside C, the Taylor series expansion of $f(z)$ about $z = a$, is

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+....+\frac{f^n(a)}{n!}(z-a)^n+....$$



$$\Rightarrow\ f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n$$

Example 33: Expand the function $f(x) = \sin x$ about the point $x=\frac{\pi}{6}$.

Soln: $$f(x)=f\left(\frac{\pi}{6}\right)+\left(x-\frac{\pi}{6}\right)f'\left(\frac{\pi}{6}\right)+\frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 f''\left(\frac{\pi}{6}\right)+......$$

$$=\sin\frac{\pi}{6}+\left(x-\frac{\pi}{6}\right)\cos\frac{\pi}{6}+\frac{\left(x-\frac{\pi}{6}\right)^2}{2!}\left(-\sin\frac{\pi}{6}\right)+........$$

$$=\frac{1}{2}+\left(x-\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)^2\frac{1}{2}-\frac{1}{6}\left(x-\frac{\pi}{6}\right)^3\frac{\sqrt{3}}{2}+......$$

$$f(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{1}{4\sqrt{3}}\left(x-\frac{\pi}{6}\right)^3+....$$