First excited state (n=1): $\psi_1(x) = \left(\dfrac{\alpha}{2\sqrt{\pi}}\right)^{1/2} (2\alpha x) e^{-\alpha^2 x^2/2}$

Second excited state (n=2): $\psi_2(x) = \left(\dfrac{\alpha}{8\sqrt{\pi}}\right)^{1/2} (4x^2\alpha^2 - 2) e^{-\alpha^2 x^2/2}$

Third excited state (n=3): $\psi_3(x) = \left(\dfrac{\alpha}{48\sqrt{\pi}}\right)^{1/2} (8\alpha^3 x^3 - 12\alpha x) e^{-\alpha^2 x^2/2}$

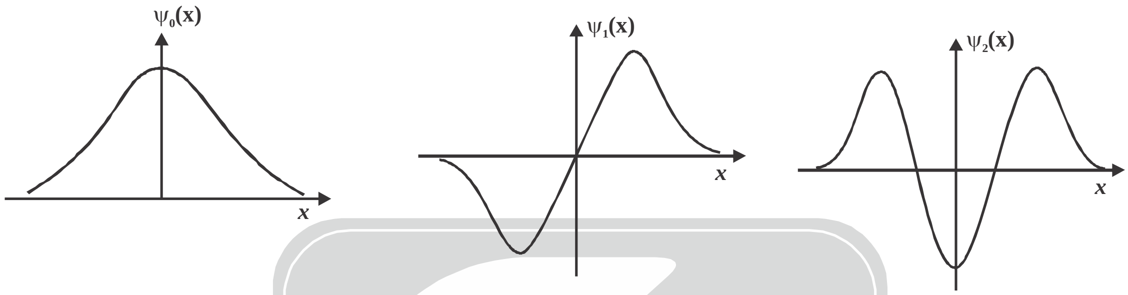


Figure : Schematic diagram of the wavefunction of the particle in ground, first excited and second excited state respectively

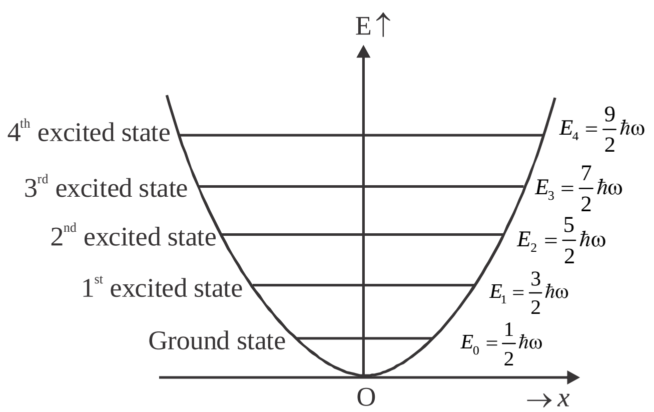
Note: (i) $\psi_n(x)$ is an even function if n = even and is an odd function if n = odd.

(ii) $\psi_n(x)$ has 'n' no. of nodes.

Energy eigenvalue of a particle moving under linear harmonic oscillator potential will be

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad [n = 0, 1, 2, 3.......]$$

It is an infinite sequence of discrete energy levels with equal spacing of $\hbar\omega$. For n = 0, we get $E_0 = \dfrac{1}{2}\hbar\omega$ and it is known as zero point energy of the oscillator. The existance of zero point energy is in accordance with heisenburg uncertainty principle.



- **Existance of zero point energy according to Uncertainity principle:**

Let a particle of mass 'm' executes SHM along x axis and at time 't' its position is x from the mean position.

Total energy of LHO $E = E_k + E_p = \dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 x^2 = \dfrac{(\Delta p)^2}{2m} + \dfrac{1}{2}m\omega^2(\Delta x)^2$

Since, $\Delta x = x$ then $\Delta p = \dfrac{\hbar}{2\Delta x} = \dfrac{\hbar}{2x}$

Therefore, $E = \dfrac{1}{2m}\left(\dfrac{\hbar}{2x}\right)^2 + \dfrac{1}{2}m\omega^2 x^2 = \dfrac{\hbar^2}{8mx^2} + \dfrac{1}{2}m\omega^2 x^2 \Rightarrow \dfrac{dE}{dx} = -\dfrac{\hbar^2}{4mx^3} + m\omega^2 x$

Putting $\dfrac{dE}{dx} = 0 \Rightarrow x^2 = \dfrac{\hbar}{2m\omega}$

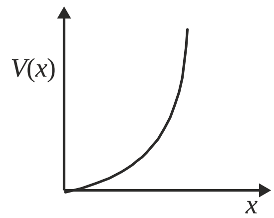
Therefore, the miminum energy of the oscillator

$$E_{\min} = \frac{\hbar}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega$$

- **Half-Harmonic Oscillator:**

Consider half-harmonic oscillator potential i.e.

$$V(x) = \begin{cases} \infty & x < 0 \\ \dfrac{1}{2}m\omega^2 x^2 & x > 0 \end{cases}$$



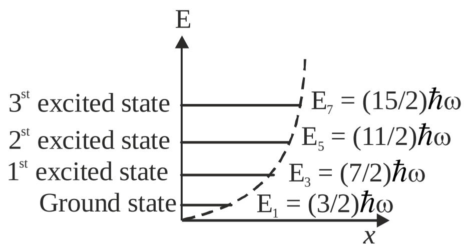
A physical interpretation of this could be a spring that can be stretched from its equilibrium position but not compressed.

We can find the allowed energies of this potential by considering its difference from the ordinary harmonic oscillator. In the ordinary case, there were no boundary conditions, and we found the stationary states could be expressed in terms of the Hermite polynomials.

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{\sqrt{m\omega}}{\hbar}x\right)e^{-m\omega x^2/2\hbar}$$

The Hermite polynomials are even if n is even and odd if n is odd. Since all the even Hermite polynomials have no constant term, $H(0) = 0$ if n is odd

From continuity of the wave function at $x = 0$ we must have $\psi(0) = 0$ (since the wave function is zero for $x < 0$). The solution above still aplies for $x > 0$, but due to the boundary condition, we are allowed only the odd Hermite polynomial solutions for $x > 0$, which in turn means that the allowed energies are



$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

for n odd only.