Plane Wave solutions:

The solution of the equations (5) and (6) are plane wave.

$$\left.\begin{aligned}\vec{E}(r,t) &= \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\ \vec{H}(r,t) &= \vec{H}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{aligned}\right\} \quad \ldots (23)$$

$\vec{E}_0, \vec{H}_0$ are the complex amplitude, and $\vec{k}$ is the propagating constant. $\vec{k}$ is defined as

$$k = \frac{2\pi}{\lambda}\hat{k} = \frac{\omega}{c}\hat{k} \qquad (\hat{k} \text{ is unit vector along propagation direction})$$

Relative direction of $\vec{E}, \vec{H}, \vec{k}$

We have solution of wave equation,

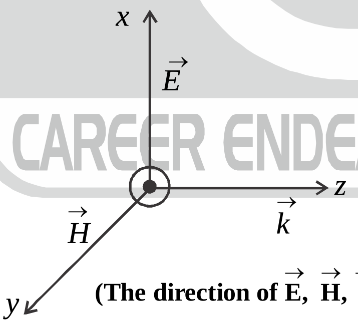
$$\vec{E}(r,t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{H}(r,t) = \vec{H}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Now,

$$\left.\begin{aligned}
&\vec{\nabla}\cdot\vec{E} = 0 && \Rightarrow i\vec{k}\cdot\vec{E} = 0 && \therefore \vec{k}\perp\vec{E}\\
&\vec{\nabla}\cdot\vec{H} = 0 && \Rightarrow i\vec{k}\cdot\vec{H} = 0 && \therefore \vec{k}\perp\vec{H}\\
&\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} && \Rightarrow i\vec{k}\times\vec{E} = i\omega\mu_0\vec{H} && \therefore \vec{k}\times\vec{E} = \omega\mu_0\vec{H}\\
&\vec{\nabla}\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} && \Rightarrow i\vec{k}\times\vec{H} = -i\omega\varepsilon_0\vec{E} && \therefore \vec{k}\times\vec{H} = \omega\varepsilon_0\vec{E}
\end{aligned}\right\} \quad \ldots. (24)$$

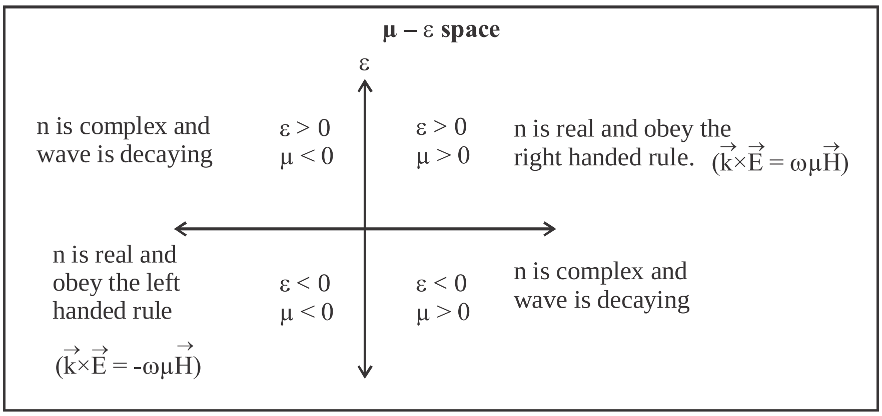
From equation we can say $\vec{k}, \vec{E}, \vec{H}$ are mutually perpendicular to each other.

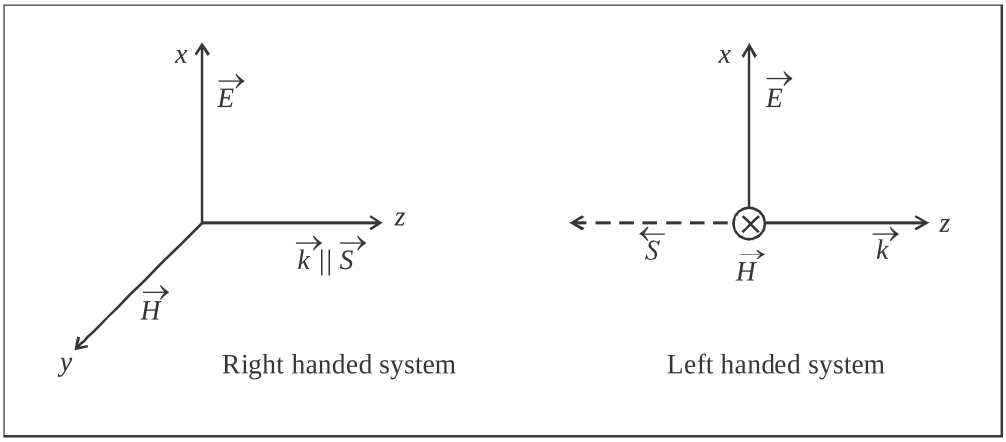


(The direction of $\vec{E}$, $\vec{H}$, $\vec{k}$) in vacuum

Special Note :

Right handed and Left handed systems :





Relative phase of $\vec{E}$ and $\vec{H}$:

We have the wave equation $\nabla^2 E - \varepsilon_0 \mu_0 \dfrac{\partial^2 E}{\partial t^2} = 0$ and the solution, $\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

So, we can write,

$$k^2 - \varepsilon_0 \mu_0 \omega^2 = 0$$

$$k^2 = \varepsilon_0 \mu_0 \omega^2 \qquad \text{... (25)}$$

Therefore, k is real quantity in free space and $\vec{E}$ and $\vec{H}$ are in phase.

Wave Impedance :

Let z_0 is a quantity of the ratio of the magnitude of $\vec{E}$ and $\vec{H}$ is

$$[z_0] = \frac{[VL^{-1}]}{[IL^{-1}]} = \frac{[V]}{[I]} = [R]$$

$$z_0 = \frac{|\vec{E}|}{\vec{H}} = \frac{\mu_0 \omega}{k} = \sqrt{\frac{\mu_0}{\in_0}} \quad \text{[Using equation (24)]}$$

$\mu_0 = 4\pi \times 10^{-7}$ $\quad S.I.$ unit

$\varepsilon_0 = 8.85 \times 10^{-12}$ $\quad S.I.$ unit

$z_0 = 376.6\,\Omega$

z_0 is called the empidence of wave in free space.

Poynting vector :

The poynting's vector for the plane electromagnetic wave in free space is

$$\vec{s} = \vec{E} \times \vec{H} = \frac{1}{\mu_0 \omega} \vec{E} \times (\vec{k} \times \vec{E}) \qquad \text{[Using equation (24)]}$$

$$= \frac{1}{\mu_0 \omega}\left[\vec{k}(\vec{E}\cdot\vec{E}) - \vec{E}(\vec{E}\cdot\vec{k})\right] = \frac{E^2}{\mu_0 \omega}\vec{k} \qquad \text{... (26)}$$

Thus, the energy flow is in the direction of wave propagation. Since $\vec{E}$ is normal to $\vec{k}$, from equation (9), we can write an terms of magnitude,

$$kE = \mu_0 \omega H \quad or \quad \sqrt{\epsilon_0} E = \sqrt{\mu_0} H \quad or \quad \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \mu_0 H^2 \qquad \text{... (28)}$$

This shows that in case of electromagnetic waves in free space electromagnetic energy is equally shared between electric and magnetic fields.

Total electromagnetic energy density is

$$u = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 = \epsilon_0 E^2 \qquad \text{... (29)}$$

So, equation (11) can also be written as

$$\boxed{\vec{s} = uc\hat{n}} \qquad \text{... (14)}$$

where, $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{\omega}{k}$ is the speed of the wave.

Time average Poynting's vector:

$$\langle \vec{s} \rangle = \frac{1}{2} \text{Re}\left[\vec{E}^* \times \vec{H} \right]$$

The magnitude of the time average of the Poynting's vector is called the intensity of radiation (I). Thus, the intensity.

$$I = |\langle s \rangle| = \frac{1}{2} E_0 H_0 = \frac{E_0}{\sqrt{2}} \frac{H_0}{\sqrt{2}} = E_{rms} \times H_{rms}$$

We also know, $\langle \mu \rangle = \langle \varepsilon_0 E_0^2 \rangle = \frac{1}{2} \varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2$

Since, Maxwell's equation is symmetric of electric field and magnetic field.

So, we can write,

$$\sqrt{\varepsilon_0} E_{rms} = \sqrt{\mu_0} H_{rms}$$

$$\therefore \quad \frac{I}{\langle u \rangle} = \frac{1}{\varepsilon_0} \cdot \frac{H_{rms}}{E_{rms}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c$$

or $$\boxed{I = \langle u \rangle c} \qquad \text{... (30)}$$

Example: An electromagnetic wave in free space is described by $\vec{E}(x, y, z, t) = \hat{z} E_0 \cos \frac{1}{2}(kx - \sqrt{3}ky - 2\omega t)$.

The Poynting vector associated with this wave is along the direction **[TIFR 2017]**

(a) $\hat{x} + \sqrt{3}\,\hat{y}$ (b) $\sqrt{3}\,\hat{x} + \hat{y}$ (c) $-\sqrt{3}\,\hat{x} + \hat{y}$ (d) $\hat{x} - \sqrt{3}\,\hat{y}$

Soln. Direction of poynting vector is along $\hat{k}$.

We have, $\frac{k_x}{2} - \frac{\sqrt{3}k}{2} y = k_x x + k_y y = \vec{k} \cdot \vec{r}$

where $\vec{k} = k_x \hat{x} + k_y \hat{y}$, $\vec{r} = x\hat{x} + y\hat{y}$

$$\therefore \quad \vec{k} = \frac{k}{2}\hat{x} - \frac{\sqrt{3}k}{2}\hat{y}$$

$$\vec{k}=\frac{1}{\left(\frac{k^2}{4}+\frac{3k^2}{4}\right)^{1/2}}\left[\frac{k}{2}\hat{x}-\frac{\sqrt{3}k}{2}\hat{y}\right]$$

$$=\frac{1}{k\left(\frac{1}{4}+\frac{3}{4}\right)^{1/2}}\left[\frac{k}{2}\hat{x}-\frac{\sqrt{3}k}{2}\hat{y}\right]$$

$$=\frac{2}{k}\left[\frac{k}{2}\hat{x}-\frac{\sqrt{3}k}{2}\hat{y}\right]=\hat{x}-\sqrt{3}\hat{y}$$

Correct option is (d)

Plane Electromagnetic Waves in an isotropic Dielectric Medium:

Maxwell's equation for a Homogeneous, non-conducting dielectric are

$$\left.\begin{aligned}&\vec{\nabla}\cdot\vec{E}=0\\&\vec{\nabla}\cdot\vec{H}=0\\&\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}\\&\vec{\nabla}\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}\end{aligned}\right\}\qquad\ldots(31)$$

All the quantities will be same like wave in free space only ε_0 replace by ε.

Electromagnetic waves in Plasma :

Plasma is quasineutral gas of charged and neutral particles which exhibits collective behaviour. Here, we neglect the motion of ions because it is massive as compared to the electrons.

Let $\vec{E}=\vec{E}(r)e^{-i\omega t}$ is the incident field.

Therefore, the equation motion of electron under the action of the incident electromagnetic field is

$$m\frac{d\vec{v}}{dt}=e\vec{E}\qquad\ldots(32)$$

$$\therefore\quad v=\frac{e\vec{E}(r)}{-i\omega m}=\frac{ieE(r)}{m\omega}$$

If n_0 be the electron density inside the plasma.

$$\vec{J}=n_0e\vec{V}=\frac{in_0e^2}{m\omega}E(r)=\sigma\,\vec{E}\qquad\text{where, }\sigma=\frac{in_0e^2}{m\omega}$$

The Maxwell's equation in plasma,

$$\left.\begin{aligned}&\vec{\nabla}\cdot\vec{E}=\frac{(n_i-n_e)e}{\varepsilon_0}=0\quad(n_i\simeq n_e=n_0)\\&\vec{\nabla}\cdot\vec{H}=0\\&\vec{\nabla}\times\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t}\\&\vec{\nabla}\times\vec{H}=J+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\end{aligned}\right\}\qquad\ldots(33)$$