• $\left[\hat{S}^2, \hat{S}_z\right] = 0$ i.e. they have simultaneous eigenstate as following:

$$\hat{S}^2 \left| s, m_s \right\rangle = s(s+1)\hbar^2 \left| s, m_s \right\rangle$$

$$\hat{S}_z \left| s, m_s \right\rangle = m_s \hbar \left| s, m_s \right\rangle$$

• Raising and lowering operators are defined as following:

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$$

$$\hat{S}_+ \left| s, m_s \right\rangle = \hbar\sqrt{(s-m_s)(s+m_s+1)} \left| s, m_s+1 \right\rangle$$

$$\hat{S}_- \left| s, m_s \right\rangle = \hbar\sqrt{(s+m_s)(s-m_s+1)} \left| s, m_s-1 \right\rangle$$

• For a spin $\frac{1}{2}$ particles like electrons, m_s can take values $\frac{1}{2}$ and $-\frac{1}{2}$, so the possible spin states are

Spin-up state: $\chi_{1/2} = \left|\uparrow\right\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$

Spin-down state: $\chi_{-1/2} = \left|\downarrow\right\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$

such that

$$\hat{S}^2 \chi_{1/2} = \frac{3}{4}\hbar^2 \chi_{1/2}, \quad \hat{S}^2 \chi_{-1/2} = \frac{3}{4}\hbar^2 \chi_{-1/2}$$

$$\hat{S}_z \chi_{1/2} = \frac{\hbar}{2}\chi_{1/2}, \quad \hat{S}_z \chi_{-1/2} = \frac{\hbar}{2}\chi_{-1/2}$$

$$\hat{S}_+ \chi_{1/2} = 0, \quad \hat{S}_+ \chi_{-1/2} = \hbar\chi_{1/2}$$

$$\hat{S}_- \chi_{1/2} = \hbar\chi_{-1/2}, \quad \hat{S}_- \chi_{-1/2} = 0$$

Matrix representation of the various spin operators for a spin-½ particle:

$$\hat{S}^2 = \begin{bmatrix} \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}^2 \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}^2 \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}^2 \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}^2 \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{bmatrix} = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{S}_z = \begin{bmatrix} \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}_z \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}_z \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}_z \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}_z \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{S}_+ = \begin{bmatrix} \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}_+ \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}_+ \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}_+ \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}_+ \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{bmatrix} = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix} = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{S}_- = \begin{bmatrix} \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}_- \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, \frac{1}{2} \middle| \hat{S}_- \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}_- \middle| \frac{1}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2} \middle| \hat{S}_- \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{bmatrix} = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\hat{S}_x = \frac{1}{2}\left[\hat{S}_+ + \hat{S}_-\right] = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{S}_y = \frac{1}{2i}\left[\hat{S}_+ - \hat{S}_-\right] = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Eigenvalues and eigenvectors of $\hat{S}_z$:

Eigenvalues: $\lambda = \pm\frac{\hbar}{2}$

Eigenvectors: $\chi_{1/2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda = \frac{\hbar}{2}$ and $\chi_{-1/2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $\lambda = -\frac{\hbar}{2}$

Eigenvalues and eigenvectors of $\hat{S}_x$:

Eigenvalues: $\lambda = \pm\frac{\hbar}{2}$

Eigenvectors: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left[\chi_{1/2} + \chi_{-1/2}\right]$ for $\lambda = \frac{\hbar}{2}$

and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left[\chi_{1/2} - \chi_{-1/2}\right]$ for $\lambda = -\frac{\hbar}{2}$

Eigenvalues and eigenvectors of $\hat{S}_y$:

Eigenvalues: $\lambda = \pm\frac{\hbar}{2}$

Eigenvectors: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\left[\chi_{1/2} + i\chi_{-1/2}\right]$ for $\lambda = \frac{\hbar}{2}$

and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}}\left[\chi_{1/2} - i\chi_{-1/2}\right]$ for $\lambda = -\frac{\hbar}{2}$

- **Pauli Spin Matrices:**

Spin angular momentum can be related with Pauli spin vector as follows:

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are components of the Pauli spin vectors and known as Pauli spin matrices. These matrices satisfies the following conditions:

(i) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

(ii) $\sigma_j \sigma_k + \sigma_k \sigma_j = \{\sigma_j, \sigma_k\} = 2I\delta_{jk}$

(iii) $[\sigma_j, \sigma_k] = 2i\,\varepsilon_{jkl}\sigma_l$

(iv) Pauli spin matrices are hermitian, traceless and determinant is -1.

(v) $\sigma_x \sigma_y \sigma_z = iI$

(vi) Since spin does not depend on spatial degrees of freedom then the components of spin S_x, S_y, S_z commute with all spatial operators i.e. momentum, position, orbital angular momentum.

(vii) $e^{i\alpha\sigma_j} = I\cos\alpha + i\sigma_j \sin\alpha$

(viii) For any two vectors $\vec{A}$ and $\vec{B}$, $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = (\vec{A}.\vec{B})I + i\sigma.(\vec{A}\times\vec{B})$

Example 6. The wave function of an electron at an instant is given by $\psi = f(r,\theta)e^{2i\phi}\chi_{1/2}$. Calculate the average value of z-component of its magnetic moment.

Soln. The operator for the z-component of the magnetic moment is

$$\mu_z = \mu_{\ell z} + \mu_{sz} = \frac{\mu_B}{\hbar}L_z + 2\frac{\mu_B}{\hbar}S_z$$

The average value of μ_z in the given state is $\langle \mu_z \rangle = \langle \psi | \mu_z | \psi \rangle$

Now, $$\mu_z|\psi\rangle = \frac{\mu_B}{\hbar}\left[L_z|\psi\rangle + 2S_z|\psi\rangle\right]$$

The given wave function is $\psi = f(r,\theta)e^{2i\phi}\chi_{1/2}$. The ϕ dependent part is $e^{2i\phi}$. This shows that it is an eigenfunction of L_z with eigenvlaue $2\hbar$. The spin dependent part is $\chi_{1/2}$. This shows that it is an eigenfunction of S_z with eigenvalue $+\frac{\hbar}{2}$. Thus

$$L_z|\psi\rangle = 2\hbar|\psi\rangle \text{ and } S_z|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$$

Therefore, $$\mu_z|\psi\rangle = \frac{\mu_B}{\hbar}\left[2\hbar|\psi\rangle + 2\frac{\hbar}{2}|\psi\rangle\right]$$

So, $$\langle\psi|\mu_z|\psi\rangle = \mu_B\left[2\langle\psi|\psi\rangle + \langle\psi|\psi\rangle\right] = 3\mu_B$$

Example 7. The spin part of the wave function of a spin-½ particle is $\cos\alpha\ \chi_{1/2} + \sin\alpha\ e^{i\beta}\chi_{-1/2}$. The *x*-component of the spin is measured. Find the probability of getting the result $\frac{\hbar}{2}$. Here α and β are real constants.

Soln. The wave function before the measurement is $|\chi_1\rangle = \cos\alpha_{\chi_{1/2}} + \sin\alpha\ e^{i\beta}\chi_{-1/2}$.

The wave function after the measurement will be eigenfunction of S_x corresponding to the eigenvalue $\frac{\hbar}{2}$ i.e.

$$|\chi\rangle = \frac{1}{\sqrt{2}}(\chi_{1/2} + \chi_{-1/2})$$