Chapter 2

Stability Analysis

"Nothing takes place in the world whose meaning is not that of some maximum or minimum" **Leonhard Euler**

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

2.1 Equilibrium criteria in one dimension:

If $V(x)$ be the potential under which a particle is moving then force acting on the particle is $F_x = -\frac{dV(x)}{dx}$.

At equilibrium point $F_x = 0$. Therefore, condition for equilibrium in terms of potential becomes, $\frac{dV(x)}{dx} = 0$

In $V(x)$ versus x graph, $\frac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to x axis. Therefore in the figure shown below point A, B, C and D are equilibrium points.

Stable equilibrium point: A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice versa. That is the force tries to bring it back to the point of equilibrium.

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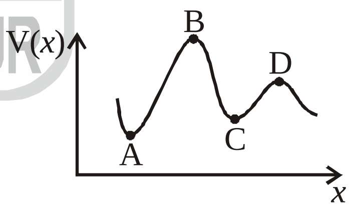
Therefore at stable equilibrium point $\frac{dF_x}{dx} < 0$



$$\therefore \frac{d^2V(x)}{dx^2} > 0 \text{ (condition for minimum)}$$

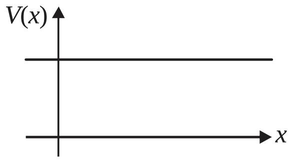
Thus, a stable equilibrium point is a minimum on $V(x)$ versus x plot.
Therefore points A and C in the figure shown above are stable equilibrium point.

Unstable equilibrium point: In this case, if a particle at equilibrium point is displaced towards right, it experiences a force that is directed to rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point, $\frac{dF_x}{dx} > 0$

$$\therefore \frac{d^2V(x)}{dx^2} < 0 \text{ (condition of maximum)}$$

Thus, an unstable point is a maximum on $V(x)$ versus x plot. Therefore points B and D in the figure shown above are unstable points.

Neutral equilibrium: If the potential energy of a particle is constant irrespective of its position, then it is said to be in neutral equilibrium.



Frequency of oscillation: We know a particle of mass m moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point. For a particle moving under some arbitrary potential $V(x)$, if x_0 be the stable equilibrium point, then we can expand $V(x)$ about x_0 using Taylor's series expansion as,

$$V(x) = V(x_0) + (x - x_0)\frac{dV}{dx}\bigg|_{x=x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2V}{dx^2}\bigg|_{x=x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3V}{dx^3}\bigg|_{x=x_0} + \cdots$$

where $(x - x_0)$ is displacement from stable equilibrium point.

At this point, $\frac{dV}{dx}\bigg|_{x=x_0} = 0$, therefore, if $(x - x_0)$ be small then

$$V(x) = V(x_0) + \frac{(x-x_0)^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0} \quad \text{(higher power terms neglected)}$$

or $\quad V(x) = V(x_0) + \frac{1}{2}k(x - x_0)^2$

Therefore frequency of oscillation about stable equilibrium is

$$\omega = \sqrt{\frac{k}{m}} \quad \text{where,} \quad k = \frac{d^2V}{dx^2}\bigg|_{x=x_0}$$, k is called force constant.

2.2 Equilibrium criteria in two dimension: If a particle is moving under a two dimensional potential $V(x, y)$ then, at equilibrium point,

$$\frac{\partial}{\partial x}V(x,y)\bigg|_{x_0,y_0} = 0, \quad \frac{\partial}{\partial y}V(x,y)\bigg|_{x_0,y_0} = 0$$

for stable equilibrium point (minimum)

$$\frac{\partial^2}{\partial x^2}V(x,y)\bigg|_{x_0,y_0} > 0, \quad \frac{\partial^2}{\partial y^2}V(x,y)\bigg|_{x_0,y_0} > 0$$

$$\frac{\partial^2}{\partial x^2}V(x,y)\bigg|_{x_0,y_0} \cdot \frac{\partial^2}{\partial y^2}V(x,y)\bigg|_{x_0,y_0} > \left[\frac{\partial^2}{\partial x \partial y}V(x,y)\bigg|_{x_0,y_0}\right]^2$$

For unstable equilibrium point (maximum)