Chapter 4

Electrostatic Energy

Electrostatic Energy:

The work necessary to assemble a system of charges against coulomb forces is stored in the field as potential energy. This is known as electrostatic energy.

Electrostatic energy system of point charges:

Let at any time we have placed a point charge q_1 at $\vec{r}_1$. To place the charge q_1 at the position $\vec{r}_1$, we require no work because there is no interacting coulomb field.

$\therefore \qquad U_1 = 0$

But to bring the charge q_2 to the position $\vec{r}_2$ we require work done against coulomb repulsion due to q_1.

This work $\qquad U_2 = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r_{12}} \frac{q_1 q_2}{r^2} dr = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|r_1 - r_2|} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$

Therefore for two point charges,

$$\boxed{u = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}}$$

To bring another charge, we need work done.

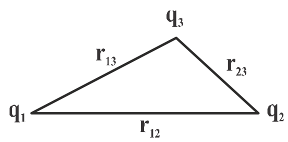
$$U_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Therefore, net work done

$$U = U_1 + U_2 + U_3 = 0 + \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

For N point charged,

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}}$$



We know electrostatic potential

$$\phi_i = \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}$$

$\therefore \quad \boxed{U = \frac{1}{2}\sum_{i=1}^{N} q_i \phi_i}$

Potential energy of interaction of a charge 'q' with other charges

$u = qV$, where V is potential at location of q due to other charges.

For continuous charge distribution :

For continous charge distribution u is given by

$$\boxed{U = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\, dr} \qquad [\,\rho(\vec{r}) \text{ is volume charge distribution}]$$

Electrostatic Energy in terms of field :

Suppose we have a finite region of space V in a dielectric medium of permitivity ε and the volume charge density ρ .

Therefore, the electrostatic energy of the system is given by

$$U = \frac{1}{2}\varepsilon_0 \int \rho(r)\phi(r)\, dV = \frac{1}{2}\int \left(\vec{\nabla}\cdot\vec{E}\right)\phi\, dV \qquad [\text{Gauss's law } \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho\,]$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\vec{\nabla}\cdot\phi\vec{E}\right) - \vec{\nabla}\phi\cdot\vec{E}\Big)\, dV$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\vec{\nabla}\cdot\phi\vec{E}\right) dV - \frac{\varepsilon_0}{2}\int \left(\vec{\nabla}\phi\cdot\vec{E}\right) dV$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\phi\vec{E}\right) dS + \frac{\varepsilon_0}{2}\left(\vec{E}\cdot\vec{E}\right) dV$$

At large distance from the charge distribution the first term will be vanish.

$\therefore \quad \boxed{U = \frac{1}{2}\varepsilon_0 \int E^2 dV}$

Therefore, the energy density, $\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$

Forces and Torques from electrostatic energy:

We know work done = force . displacement

$\Rightarrow \quad dW = \vec{F}\cdot d\vec{r}$

Also for isolated system, the work done

$$dW = -dU = -\vec{\nabla}U\cdot d\vec{r}$$

$\therefore \quad \boxed{\vec{F} = -\vec{\nabla}U}\,,\ u = -\int \vec{F}\cdot d\vec{r}$

If the part of the system under consideration is constrained to rotate about an axis and it undergoes a small rotation do under the action of a torque $\vec{\tau}$ then,

$$dW = \vec{\tau}\cdot d\vec{\theta} = -dU = -\nabla_\theta U \cdot d\theta$$

$\therefore \quad \boxed{\vec{\tau} = -\nabla_\theta U} \quad \text{and } u = \int \vec{F}\cdot d\vec{r}$