Chapter 4

Angular Momentum and Spin

4.1. Orbital angular momentum :

In classical mechanics, the angular momentum $\vec{L}$ of a particle is defined as

$$\vec{L}=\vec{r}\times\vec{p}=\left(yp_z-zp_y\right)\hat{i}+\left(zp_x-xp_z\right)\hat{j}+\left(xp_y-yp_x\right)\hat{k}$$

Replacing the position and momentum co-ordinates by the equivalent quantum mechanical operators, we can have

	Cartesian co-ordinates	Spherical Polar Co-ordiantes
$\hat{L}_x$	$-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$	$i\hbar\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$
$\hat{L}_y$	$-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)$	$i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$
$\hat{L}_z$	$-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$	$-i\hbar\frac{\partial}{\partial\phi}$
$\hat{L}^2$		$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$

Commmutation Relations

$$\left[\hat{L}_x,\hat{L}_y\right]=i\hbar\hat{L}_z \qquad \left[\hat{L}_y,\hat{L}_z\right]=i\hbar\hat{L}_x \qquad \left[\hat{L}_z,\hat{L}_x\right]=i\hbar\hat{L}_y$$

$$\left[\hat{L}_y,\hat{L}_x\right]=-i\hbar\hat{L}_z \qquad \left[\hat{L}_z,\hat{L}_y\right]=-i\hbar\hat{L}_x \qquad \left[\hat{L}_x,\hat{L}_z\right]=-i\hbar\hat{L}_y$$

i.e. the components of the orbital angular momentums cannot be measured simultaneously accurately.

$$\left[\hat{L}^2,\hat{L}_x\right]=0 \qquad \left[\hat{L}^2,\hat{L}_y\right]=0 \qquad \left[\hat{L}^2,\hat{L}_z\right]=0$$

i.e. square of the orbital angular momentum commutes with any one of components of the orbital angular momentum.

$$\left[\hat{L}_x,\hat{p}_x\right]=0 \qquad \left[\hat{L}_y,\hat{p}_y\right]=0 \qquad \left[\hat{L}_z,\hat{p}_z\right]=0$$

$$\left[\hat{L}_x, \hat{p}_y\right] = i\hbar\hat{p}_z \qquad \left[\hat{L}_y, \hat{p}_z\right] = i\hbar\hat{p}_x \qquad \left[\hat{L}_z, \hat{p}_x\right] = i\hbar\hat{p}_y$$

$$\left[\hat{L}_x, \hat{x}\right] = 0 \qquad \left[\hat{L}_y, \hat{y}\right] = 0 \qquad \left[\hat{L}_z, \hat{z}\right] = 0$$

$$\left[\hat{L}_x, \hat{y}\right] = i\hbar\hat{z} \qquad \left[\hat{L}_y, \hat{z}\right] = i\hbar\hat{x} \qquad \left[\hat{L}_z, \hat{x}\right] = i\hbar\hat{y}$$

Eigevalues and eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$:

Since, the commutator bracket $\left[\hat{L}^2, \hat{L}_z\right] = 0$, therefore $\hat{L}^2$ and $\hat{L}_z$ can be measured simultaneously accurately and both have simultaneous eigenfunctions or eigenkets. Denoting the simultaneous eigenkets by $|l, m_l\rangle$ (where l and m_l are the orbital quantum number and orbital magnetic quantum number respectively), the eigenvalue equation for $\hat{L}^2$ and $\hat{L}_z$ can be written as

$$\hat{L}^2 |l, m_l\rangle = l(l+1)\hbar^2 |l, m_l\rangle$$

$$\hat{L}_z |l, m_l\rangle = m_l\hbar |l, m_l\rangle$$

Alternate form: Since, $\hat{L}^2$ and $\hat{L}_z$ depends on θ, ϕ then their eigen function is function of θ, ϕ i.e.

$$\hat{L}^2 Y_{\ell,m_l}(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_{\ell,m_l}(\theta,\phi)$$

$$\hat{L}_z Y_{\ell,m_l}(\theta,\phi) = m_l\hbar Y_{\ell,m_l}(\theta,\phi)$$

where $Y_{\ell,m}(\theta,\phi)$ is said to be Spherical Harmonics and defined as

$$Y_{l,m_l}(\theta,\varphi) = \varepsilon\sqrt{\left(\frac{2\ell+1}{4\pi}\right)\frac{(\ell-|m_l|)!}{(\ell+|m_l|)!}}\, P_\ell^{|m_l|}(\cos\theta)\, e^{im_l\phi}$$

where $\varepsilon = (-1)^{m_l}$ for $m_l > 0$ and $\varepsilon = 1$ for $m_l \le 0$.

Raising and Lowering Operators:

Raising Operator: $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$

Lowering Operator: $\hat{L}_- = \hat{L}_x - i\hat{L}_y$

Important relations:

$$\left[\hat{L}_z, \hat{L}_+\right] = \hbar\hat{L}_+ \qquad \left[\hat{L}_z, \hat{L}_-\right] = -\hbar\hat{L}_- \qquad \left[\hat{L}_x, \hat{L}_+\right] = -\hbar\hat{L}_z \qquad \left[\hat{L}_x, \hat{L}_-\right] = \hbar\hat{L}_z$$

$$\left[\hat{L}_y, \hat{L}_+\right] = -i\hbar\hat{L}_z \qquad \left[\hat{L}_y, \hat{L}_-\right] = i\hbar\hat{L}_z \qquad \left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z$$

$$\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z \qquad \hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$$

Action of $\hat{L}_+$ and $\hat{L}_-$:

$$\hat{L}_+ |l, m_l\rangle = \sqrt{(\ell - m_l)(l + m_l + 1)}\hbar\, |l, m_l + 1\rangle$$

i.e. $$\hat{L}_+ Y_{\ell,m_l}(\theta,\phi) = \sqrt{(\ell - m_l)(l + m_l + 1)}\, \hbar Y_{\ell,m_l+1}(\theta,\phi)$$

$$\hat{L}_{-}|l, m_l\rangle = \sqrt{(\ell + m_l)(l - m_l + 1)}\hbar\,|l, m_l - 1\rangle$$

i.e. $$\hat{L}_{-}Y_{\ell,m_l}(\theta,\phi) = \sqrt{(\ell + m_l)(l - m_l + 1)}\,\hbar Y_{\ell,m_l-1}(\theta,\phi)$$

Matrix Representation of the operators:

Elements of $\hat{L}^2 = \langle \ell', m_l'|\hat{L}^2|\ell, m_l\rangle = \ell(\ell+1)\hbar^2\,\delta_{\ell\ell'}\,\delta_{m_l m_l'}$, will be non-zero for $l = l'$ and $m_l = m_l'$ i.e.

$$\hat{L}^2 = \begin{bmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{bmatrix} = 2\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (for } l = 1\text{)}$$

Elements of $\hat{L}_z = \langle \ell', m_l'|\hat{L}_z|\ell, m_l\rangle = m_l\hbar\,\delta_{\ell\ell'}\,\delta_{m_l m_l'}$, will be non-zero for $l = l'$ and $m_l = m_l'$ i.e.

$$\hat{L}_z = \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix} = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ (for } l = 1\text{)}$$

Elements of $\hat{L}_+ = \langle \ell', m_l'|\hat{L}_+|\ell, m_l\rangle = \hbar\sqrt{(\ell - m_l)(\ell + m_l + 1)}\,\delta_{\ell\ell'}\;\delta_{m_l', m_l+1}$, will be non-zero for $l = l'$ and $m_l = m_l' + 1$ i.e.

$$\hat{L}_+ = \begin{bmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\,\hbar \\ 0 & 0 & 0 \end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ (for } l = 1\text{)}$$

Elements of $\hat{L}_+ = \langle \ell', m_l'|\hat{L}_-|\ell, m_l\rangle = \hbar\sqrt{(\ell + m_l)(\ell - m_l + 1)}\,\delta_{\ell\ell'}\;\delta_{m_l', m_l-1}$, will be non-zero for $l = l'$ and $m_l = m_l' - 1$ i.e.

$$\hat{L}_- = \sqrt{2}\,\hbar\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ (for } l = 1\text{)}$$

Expectation values in the state $|\ell, m_\ell\rangle$:

$$\langle \hat{L}_x\rangle = 0,\ \langle \hat{L}_y\rangle = 0,\ \langle \hat{L}_z\rangle = m_l\hbar$$

$$\langle \hat{L}_x^2\rangle = \langle \hat{L}_y^2\rangle = \frac{\hbar^2}{2}\left[\ell(\ell+1) - m_l^2\right],\ \langle \hat{L}_z^2\rangle = m_l^2\hbar^2$$

Example 1: Which of the following are eigenfunctions of L_z? For the cases where the function is an eigenfuntion of L_z, find the correspondence eigenvalue.

(a) $\sin\theta\, e^{i\phi}$ (b) $e^{i(\theta+\phi)}$ (c) $e^{i\theta}\sin\phi$ (d) $r^n\cos\theta$

Soln. We know, $L_z = -i\hbar \frac{\partial}{\partial\phi}$

(a) $-i\hbar \frac{\partial}{\partial\phi}\left(\sin\theta\, e^{i\phi}\right) = (-i\hbar)\sin\theta(i)e^{i\phi} = \hbar\sin\theta\, e^{i\phi}$

So, $\sin\theta\, e^{i\phi}$ is an eigenfunction of L_z and the corresponding eigenvalue is $\hbar$.

(b) $-i\hbar \frac{\partial}{\partial\phi} e^{i(\theta+\phi)} = (-i\hbar)(i)e^{i(\theta+\phi)} = \hbar e^{i(\theta+\phi)}$

So, $e^{i(\theta+\phi)}$ is an eigenfunction of L_z and the corresponding eigenvalue is $\hbar$.

(c) $-i\hbar \frac{\partial}{\partial\phi}\left(e^{i\theta}\sin\phi\right) = -i\hbar\, e^{i\theta}\cos\phi$

So, $e^{i\theta}\sin\theta$ is not eigenfunction of L_z.

(d) $-i\hbar \frac{\partial}{\partial\phi}\left(r^n\cos\theta\right) = 0 = 0.\left(r^n\cos\theta\right)$

So, $r^n\cos\theta$ is an eigenfunction of L_z and the corresponding eigenvalue is zero.

Example 2. (a) Is $\cos\theta + \sin\theta\, e^{i\phi}$ an eigenfunction of L^2 and L_z? If yes, what are the corresponding eigenvalues?

Soln. (a) We know that, $Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$ and $Y_{1,1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

Thus $\cos\theta + \sin\theta\, e^{i\phi} = c_1 Y_{1,0}(\theta,\phi) + c_2 Y_{1,1}(\theta,\phi)$ where c_1, c_2 are constant.

$$L^2\left(\cos\theta + \sin\theta\, e^{i\phi}\right) = c_1 L^2 Y_{1,0}(\theta,\phi) + c_2 L^2 Y_{1,1}(\theta,\phi)$$

$$= c_1 2\hbar^2 Y_{1,0}(\theta,\phi) + c_2 2\hbar^2 Y_{1,1}(\theta,\phi)$$

$$= 2\hbar^2\left[c_1 Y_{1,0}(\theta,\phi) + c_2 Y_{1,1}(\theta,\phi)\right] = 2\hbar^2\left[\cos\theta + \sin\theta\, e^{i\phi}\right]$$

Thus, $\cos\theta + \sin\theta\, e^{i\phi}$ is an eigenfunction of L^2 and the corresponding eigenvalue is $2\hbar^2$.

(b) $L_z\left(\cos\theta + \sin\theta\, e^{i\phi}\right) = c_1 L_z Y_{1,0}(\theta,\phi) + c_2 L_z Y_1^1(\theta,\phi) = c_1.0 + c_2\hbar Y_{1,1}(\theta,\phi) = \hbar\sin\theta\, e^{i\phi}$

Thus, $\cos\theta + \sin\theta\, e^{i\phi}$ is not an eigenfunction of L_z.

Example 3. It is given that $Y_{2,0}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$. Find the expressions for $Y_{2,1}(\theta,\phi)$.

Soln. Expressions for L_x and L_y are given in terms of θ, ϕ. Using these,

$$L_+ = L_x + iL_y = \hbar\, e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

You can write $Y_{2,0}(\theta,\phi)$ as $|2, 0\rangle$. Also, $L_+|\ell\, m\rangle = \hbar\sqrt{(\ell-m)(\ell+m+1)}|\ell\, m+1\rangle$