Chapter 4

Electrostatic Energy

Electrostatic Energy:

The work necessary to assemble a system of charges against coulomb forces is stored in the field as potential energy. This is known as electrostatic energy.

Electrostatic energy system of point charges:

Let at any time we have placed a point charge q_1 at $\vec{r}_1$. To place the charge q_1 at the position $\vec{r}_1$, we require no work because there is no interacting coulomb field.

$\therefore \qquad U_1 = 0$

But to bring the charge q_2 to the position $\vec{r}_2$ we require work done against coulomb repulsion due to q_1.

This work $\qquad U_2 = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r_{12}} \frac{q_1 q_2}{r^2} dr = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|r_1 - r_2|} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$

Therefore for two point charges,

$$\boxed{u = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}}$$

To bring another charge, we need work done.

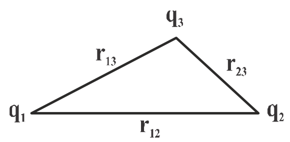
$$U_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Therefore, net work done

$$U = U_1 + U_2 + U_3 = 0 + \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

For N point charged,

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}}$$



We know electrostatic potential

$$\phi_i = \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}$$

$\therefore$ $$\boxed{U = \frac{1}{2}\sum_{i=1}^{N} q_i \phi_i}$$

Potential energy of interaction of a charge 'q' with other charges

$u = qV$, where V is potential at location of q due to other charges.

For continuous charge distribution :

For continous charge distribution u is given by

$$\boxed{U = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\,dr}$$ [$\rho(\vec{r})$ is volume charge distribution]

Electrostatic Energy in terms of field :

Suppose we have a finite region of space V in a dielectric medium of permitivity ε and the volume charge density ρ .

Therefore, the electrostatic energy of the system is given by

$$U = \frac{1}{2}\varepsilon_0 \int \rho(r)\phi(r)\,dV = \frac{1}{2}\int \left(\vec{\nabla}\cdot\vec{E}\right)\phi\,dV \qquad \left[\text{Gauss's law } \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho\right]$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\vec{\nabla}\cdot\phi\vec{E}\right) - \vec{\nabla}\phi\cdot\vec{E}\,\right) dV$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\vec{\nabla}\cdot\phi\vec{E}\right) dV - \frac{\varepsilon_0}{2}\int \left(\vec{\nabla}\phi\cdot\vec{E}\right) dV$$

$$= \frac{1}{2}\varepsilon_0 \int \left(\phi\vec{E}\right) dS + \frac{\varepsilon_0}{2}\left(\vec{E}\cdot\vec{E}\right) dV$$

At large distance from the charge distribution the first term will be vanish.

$\therefore$ $$\boxed{U = \frac{1}{2}\varepsilon_0 \int E^2 dV}$$

Therefore, the energy density, $\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$

Forces and Torques from electrostatic energy:

We know work done = force . displacement

$\Rightarrow$ $$dW = \vec{F}\cdot d\vec{r}$$

Also for isolated system, the work done

$$dW = -dU = -\vec{\nabla}U\cdot d\vec{r}$$

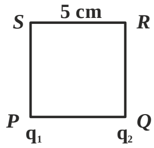
$\therefore$ $$\boxed{\vec{F} = -\vec{\nabla}U},\ u = -\int \vec{F}\cdot d\vec{r}$$

If the part of the system under consideration is constrained to rotate about an axis and it undergoes a small rotation do under the action of a torque $\vec{\tau}$ then,

$$dW = \vec{\tau}\cdot d\vec{\theta} = -dU = -\nabla_\theta U \cdot d\theta$$

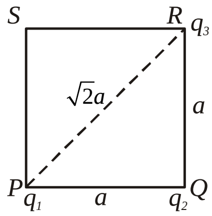
$\therefore$ $$\boxed{\vec{\tau} = -\nabla_\theta U} \quad \text{and } u = \int \vec{F}\cdot d\vec{r}$$

Example : Charges $q_1 = 2\times10^{-6}C$ and $q_2 = 1\times10^{-6}C$ are placed at the corners P and Q of a square of side 5 cm as shown in figure. The work done in moving a charge 1×10^{-6} C from corner R to S is given by



(a) 0.053 J (b) 0.53 J (c) 1.06 J (d) 0.106 J

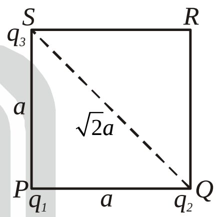
Soln. Let the third charge be $q_3 = 1\times10^{-6}c$



Initial configuration :

Potential energy of initial configuration

$$U_i = k\left[\frac{q_1q_3}{\sqrt{2}a} + \frac{q_1q_2}{a} + \frac{q_2q_3}{a}\right]$$



Final configuration :

$$U_f = k\left[\frac{q_1q_3}{a} + \frac{q_1q_2}{a} + \frac{q_3q_2}{\sqrt{2}a}\right]$$

Work done $= U_f - U_i$

$$= \frac{k}{a}\left[q_1q_3\left(1 - \frac{1}{\sqrt{2}}\right) + q_3q_2\left(\frac{1}{\sqrt{2}} - 1\right)\right]$$

$$= \frac{kq_3}{\sqrt{2}a}\left[q_1 - q_2\right]\left(\sqrt{2} - 1\right)$$

Substituting the values of $k = \frac{1}{4\pi\in_0} = 9\times10^9\, Nm^2/C^2$,

$q_1 = 2\times10^{-6}C,\ q_2 = 1\times10^{-6}C,\ a = 5\,cm,\ q_3 = 1\times10^{-6}C$

We get, work done $= \frac{9\times10^9\times1\times10^{-6}}{\sqrt{2}\times5\times10^{-2}}\left[2-1\right]\times10^{-6}\left(\sqrt{2}-1\right)$

$= 0.053\ J$

Correct option is (a)

The electrostatic energy associated with a uniform spherical charge distribution of total charge *q* and radius *R*.

The electric field of this distribution is

$$E = E_1 = \frac{qr}{4\pi\varepsilon_0 R^3} \quad \text{for } r \le R$$

$$E = E_2 = \frac{q}{4\pi\varepsilon_0 r^2} \quad \text{for } r \ge R$$

Since, the field is radially symmetric, the volume element may be expressed as $dV = 4\pi r^2 dr$, so that the energy is