1.2 Kinematics :

Instantaneous velocity $\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{r}$ = position vector

$$\therefore \quad v_x = \frac{dx}{dt} = \dot{x}, v_y = \frac{dy}{dt} = \dot{y}$$

Average velocity, $\vec{v} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$

Instantaneous acceleration, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

Average acceleration, $\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$

Uniformly accelerated motions: When a constant force acts on a particle its acceleration is constant. If u_x, v_x be initial and final velocity of a particle along x-direction and a_x be its acceleration along x direction then.

$$\left.\begin{aligned} v_x &= u_x + a_x t \\ v_x^2 &= u_x^2 + 2a_x x \\ x &= u_x t + \tfrac{1}{2} a_x t^2 \end{aligned}\right\} \text{if } a_x = \text{constant}$$

where x is displacement along x-direction and t is time taken. We can write similar relations for y and z direction also.

Non-uniformly accelerated motion: If acceleration of particle is not constant then we cannot use formulae of uniformly accelerated motion of kinematics. In that case we start with either definition of velocity or definition of acceleration i.e.

$$v_x = \frac{dx}{dt} \quad \text{or} \quad a_x = \frac{dv_x}{dt}$$

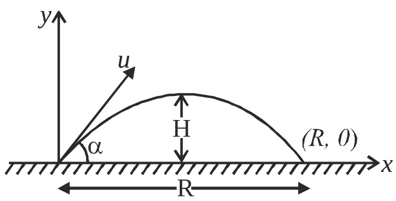
We may also have to use $a_x = \frac{F_x}{m}$ and $\frac{dv_x}{dt} = \frac{dv_x}{dx}.\frac{dx}{dt} = \frac{v_x dv_x}{dx}$ as the case may be.

Projectile motion:

It is an example of two dimensional uniformly accelerated motion.

If u be the projection speed and α be angle of projection. The components of the initial speed are

$$u_x = u\cos\alpha, u_y = u\sin\alpha$$



The components of acceleration of the particle are

$$a_x = \frac{F_x}{m} = \frac{0}{m} = 0, a_y = \frac{F_y}{m} = -\frac{mg}{m} = -g$$

Using the standard formulas for 1-D constantly accelerated motion, we have,

$$x = u_x t + \frac{1}{2} a_x t^2 = (u\cos\alpha)t \Rightarrow t = \frac{x}{u\cos\alpha} \qquad \text{... (1)}$$

Also, $y = u_y t + \frac{1}{2} a_y t^2 = u\sin\alpha t - \frac{1}{2} g t^2$ ... (2)

Velocity components of projectile at time t are,

$v_x = u_x + a_x t = u\cos\alpha,\ v_y = u_y + a_y t = u\sin\alpha - gt$

Putting t from Eq. (1) into Eq. (2), we get equation of path of projectile as

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

This implies that the equation of the projectile is a parabola.

In Eq. (2), if we put $t = T$ and $y = 0$ (coordinate of the point when projectile reaches the ground), then we get

Total time of flight, $T = \dfrac{2u\sin\alpha}{g}$

In equation (1), when we put $t = T$ and $x = R$ (coordinate of the point when projectile reaches the ground), then we get

Range, $R = \dfrac{u^2\sin 2\alpha}{g}$

It should be noted that the range of the projectile will be maximum for $\alpha = 45^\circ$, for a given initial speed u.

In Eq. (2), when we put $t = T/2$ and $y = H$ (coordinate of highest point), then we get

Maximum height, $H = \dfrac{u^2\sin^2\alpha}{2g}$.

An important relationship can be found as $\dfrac{H}{R} = \dfrac{\tan\alpha}{4}$.

Using this, the equation of the projectile can be expressed as

$$y = x\tan\alpha\left(1 - \frac{x}{R}\right) = \frac{4Hx}{R}\left(1 - \frac{x}{R}\right)$$

Note: Above results for projectile are applicable only if projectile is thrown from ground and it finally lands on ground. And projection speed is small so that projectile does not go too high. If projection speed is not small then height will be large and in that case we need to consider variation of acceleration due to gravity.

Example: If a projectile is thrown at an angle α with horizontal with initial speed u, then after what time the projectile's velocity will be perpendicular to its initial direction.

Soln. Suppose velocity after time t becomes perpendicular to initial velocity. Velocity after time t is

$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j} \quad = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$

Initial velocity $\vec{u} = u(\cos\alpha\hat{i} + \sin\alpha\,\hat{j})$

$\vec{v}$ is $\perp$ to $\vec{u}$, therefore, $\vec{v}.\vec{u} = 0 \quad \therefore u\cos^2\alpha + u\sin^2\alpha - gt\sin\alpha = 0$

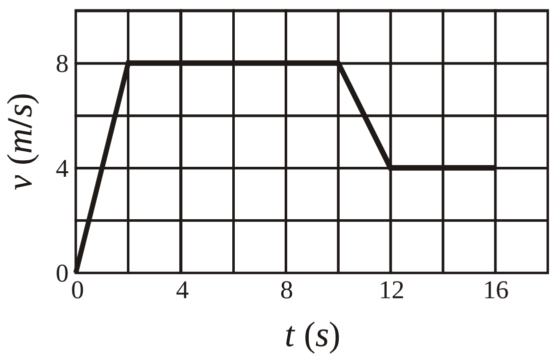
$\therefore \qquad t = \dfrac{u}{g\sin\alpha}$

Example: Consider the motion of the object whose velocity-time graph is given in the diagram.

(1) What is the acceleration of the object between times $t = 0$ and $t = 2$?

(2) What is the acceleration of the object between times $t = 10$ and $t = 12$?

(3) What is the net displacement of the object between times $t = 0$ and $t = 16$?



Soln. (1) The v-t graph is a straight line between $t = 0$ and $t = 2$, indicating constant acceleration during this time period. Hence,

$$\alpha = \frac{\Delta v}{\Delta t} = \frac{v(t=2) - v(t=0)}{2-0} = \frac{8-0}{2} = 4\ \text{ms}^{-2}.$$

(2) The v-t graph is a straight line between $t = 10$ and $t = 12$, indicating constant acceleration during this time period. Hence,

$$\alpha = \frac{\Delta v}{\Delta t} = \frac{v(t=12) - v(t=10)}{12-10} = \frac{4-8}{2} = -2\ \text{ms}^{-2}.$$

The negative sign indicates that the object is decelerating.

(3) Now, $v = \dfrac{dx}{dt}$, so $x(16) - x(0) = \int_0^{16} v(t)\,dt.$

In other words, the net displacement between times $t = 0$ and $t = 16$ equals the area under the v-t curve, evaluated between these two times. Recalling that the area of a triangle is half its width times its height, the number of grid-squares under the v-t curve is 25. The area of each grid-square is $2 \times 2 = 4$ m. Hence,

$$x(16) - x(0) = 4 \times 25 = 100\ \text{m}.$$

Example: A particle starts moving with acceleration $a = \alpha - \beta v$ where α and β are constants and v is its instantaneous speed. Find velocity of particle as a function of time and also find its terminal velocity.

Soln: Acceleration is variable therefore we start with definition of acceleration

$$a = \frac{dv}{dt} \quad \text{or} \quad \alpha - \beta v = \frac{dv}{dt} \quad \text{or} \quad \int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt$$

$$-\frac{1}{\beta} \ln\left(\frac{\alpha - \beta v}{\alpha}\right) = t \quad \text{or,} \quad 1 - \frac{\beta v}{\alpha} = e^{-\beta t} \qquad \therefore v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

As $t \to \infty$, $v = \dfrac{\alpha}{\beta}$ = constant. This is terminal velocity.

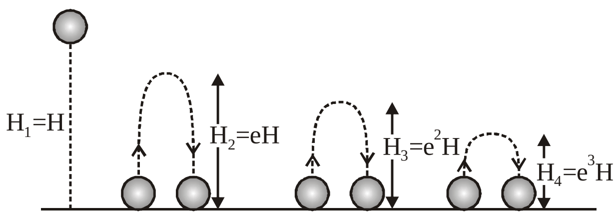
Example: A ball is dropped from a height H. It bounces back up to a height e times after hitting the ground. If $e < 1$, after what time the ball will finally come to rest.

Soln. Ball moves under the effect of gravity. Therefore magnitude of acceleration of ball during upward or downward movement remains constant.

For first bounce

$$y = H,\ u_y = 0,\ a_y = g$$

$$y = u_y t + \frac{1}{2} a_y t^2$$



$$\therefore H = 0 + \frac{1}{2}gt^2 \qquad \therefore t = \sqrt{\frac{2H}{g}}$$

Therefore time of fall before first bounce $t_1 = \sqrt{\frac{2H}{g}}$

Under gravity, time for going up a certain distance is equal to time for going down. Therefore time interval between first bounce and second bounce is

$$t_2 = 2\sqrt{\frac{2H_2}{g}} = 2\sqrt{\frac{2eH}{g}}$$

Similarly, $t_3 = 2\sqrt{\frac{2e^2H}{g}}$, $t_4 = 2\sqrt{\frac{2e^3H}{g}}$,.....

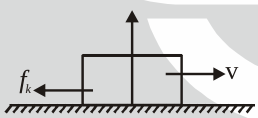
Therefore, total time elapsed before ball stops is

$$T = t_1 + t_2 + t_3 + ...$$

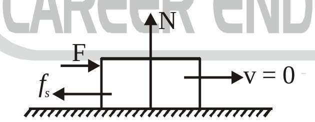
$$= \sqrt{\frac{2H}{g}}\left[1 + 2\left(e^{1/2} + e^{2/2} + e^{3/2} + ...\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2\frac{e^{1/2}}{1 - e^{1/2}}\right] = \sqrt{\frac{2H}{g}}\left(\frac{1+\sqrt{e}}{1-\sqrt{e}}\right)$$

1.3 Friction: Its arise because of interaction (electromagnetic) among the particles of surfaces in contact. It is of two types

(i) Kinetic friction: It acts on objects which are sliding on a surface $f_k = \mu_k N$

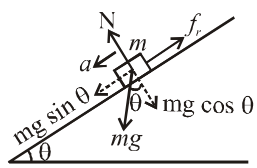


(ii) Static friction: It acts on object which are subjected to a force but not sliding on the contact surface. It is an adjustable force $f_s \le \mu_s N$



Example: A block slides down a rough inclined plane of inclination θ. Coefficient of friction between the block and the plane is μ. Calculate acceleration of block.

Soln. The block slides on the inclined plane.
Therefore, to get acceleration of block we resolve all forces parallel and perpendicular to the plane.
Therefore, for equilibrium perpendicular to the plane



$N = mg\cos\theta$ ... (i)

equation of motion parallel to plane, $mg\sin\theta - f_r = ma$ ... (ii)

and, $f_r = \mu N = \mu\, mg\cos\theta$

from (ii), $mg\sin\theta - \mu mg\cos\theta = ma \quad \therefore a = g(\sin\theta - \mu\cos\theta)$