Chapter 1

VECTOR ALGEBRA

1.1 Basic Review of Vectors

■ Definition:

Physical quantities having both magnitude and a definite direction in space. It should follow the law of vector addition.
Example: Velocity, Acceleration, Momentum, Force, Electric Field, Torque, etc.

Note: Current is a physical quantity that has both magnitude and direction but it does not follow the law of vector addition. So, current is a scalar quantity.

■ Various type of vectors:

(1) Equal vectors: Vectors having same magnitude and same direction.

(2) Null Vectors: Vectors having coincident initial and terminal point i.e. its magnitude is zero and it has any arbitrary direction.

(3) Unit Vector: Vector having unit magnitude. Unit vector along $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

(4) Reciprocal Vector: Vector having same direction as $\vec{a}$ but magnitude reciprocal to that of $\vec{a}$, is known as the reciprocal vector of $\vec{a}$. Reciprocal vector of $\vec{a}$ is $\vec{a}^{-1} = \dfrac{1}{|a|}\hat{a}$

(5) Negative Vector: Vectors having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{a}$, is known as the negative vector of $\vec{a}$. Negative vector as $\vec{a}$ is $-\vec{a} = -|a|\hat{a}$

■ Orthogonal Resolution of Vectors:

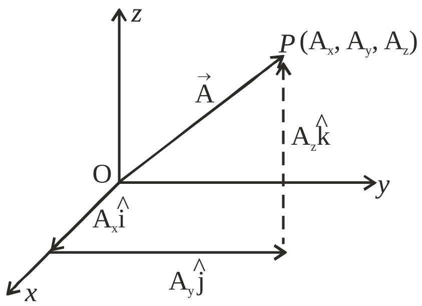
Any vector $\vec{A}$ in the 3-D right- handed rectangular cartesian coordinate system can be represented as

$$\overline{OP} = \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k},$$

where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively and A_x, A_y, A_z are the rectangular components of vector $\vec{A}$ along x, y, z axis.

Magnitude of vector $\vec{A}$ is $\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along $\vec{A}$ is $\hat{A} = \vec{A}/\left|\vec{A}\right| = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)/\sqrt{A_x^2 + A_y^2 + A_z^2}$



■ **Direction cosines of vector $\vec{A}$:**

If $\vec{A}$ makes angles α, β, γ with x, y and z axes respectively, then direction cosines of $\vec{A}$ are defined as

$$l = \cos\alpha = \frac{A_x}{A};\ m = \cos\beta = \frac{A_y}{A};\ n = \cos\gamma = \frac{A_z}{A} \quad \text{and} \quad l^2 + m^2 + n^2 = 1$$

Note: Unit vector along $\vec{A}$ can be written as $\hat{A} = l\hat{i} + m\hat{j} + n\hat{k}$

1.2 Products of Vectors

■ **Scalar Product or Dot Product:**

Dot product of two vectors $\vec{a}$ and $\vec{b}$ are defined as

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Note:

(i) Dot product is commutative in nature i.e. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(ii) For two mutually perpendicular vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0$

(iii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot i = 0$, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z$

(v) Projection of $\vec{A}$ on $\vec{B} = \vec{A}.\hat{B}$

■ **Vector Product or Cross Product :**

Cross product of two vectors $\vec{a}$ and $\vec{b}$ are defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note:

(i) Cross product is not commutative i.e. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(ii) For two collinear vectors (parallel or anti-parallel vectors) $\vec{a}\times\vec{b} = 0$.

(iii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

Example 1: The range of x for which the angle between the vectors

$$\vec{A} = 2x^2\hat{i} + 4x\hat{j} + \hat{k} \text{ and } \vec{B} = 7\hat{i} - 2\hat{j} + x\hat{k}$$

is obtuse, is equal to

(a) $x < 0$ (b) $x > 1/2$ (c) $0 < x < 1/2$ (d) None of these

Soln: Since, $90^0 < \theta < 180^0$, then $\cos\theta < 0 \Rightarrow \vec{A}.\vec{B} < 0 \Rightarrow 14x^2 - 7x < 0 \Rightarrow x(2x-1) < 0$

Case I: $x > 0$ and $(2x-1) < 0 \Rightarrow x < \frac{1}{2}$. Therefore, $0 < x < \frac{1}{2}$

Case II: $x < 0$ and $(2x-1) > 0 \Rightarrow x > \frac{1}{2}$. Both of them cannot be satisfied simultaneously.

Correct option is (c)

Example 2: Given: $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ are the sides of

(a) an isosceles triangle (b) a right-angled triangle

(c) an equilateral triangle (d) none of these

Soln: For the given set of vectors, $\vec{a}.\vec{c} = 0 \Rightarrow \vec{a}$ is perpendicular to $\vec{c}$ and

$$|\vec{a}|^2 + |\vec{c}|^2 = 14 + 21 = 35 = |\vec{b}|^2$$

Therefore, $\vec{a}$, $\vec{c}$ will correspond to base and height respectively and $\vec{b}$ will correspond to hypotenus i.e. the given set of vectors are sides of a right-angled triangle.

Correct option is (b)

Example 3: Given : $\vec{P} = 2\hat{i} - 3\hat{j}, \vec{Q} = \hat{i} + \hat{j} - \hat{k}, \vec{R} = 3\hat{i} - \hat{k}$, the vector $\vec{V}$, which is orthogonal to $\vec{P}$ and $\vec{Q}$ and having unit scalar product with $\vec{R}$, can be estimated to be

(a) $(3\hat{i} + 2\hat{j} + 5\hat{k})/4$ (b) $(3\hat{i} + 2\hat{j} - 5\hat{k})/4$ (c) $(3\hat{i} - 2\hat{j} + 5\hat{k})/4$ (d) $(3\hat{i} - 2\hat{j} - 5\hat{k})/4$

Soln: Unit vector along $\vec{V}$ is $\hat{V} = \dfrac{\vec{P}\times\vec{Q}}{|\vec{P}\times\vec{Q}|} = \dfrac{3\hat{i} + 2\hat{j} + 5\hat{k}}{\sqrt{38}}$

According to the question, $\vec{V}.\vec{R} = 1 \Rightarrow (V\hat{V}).\vec{R} = 1 \Rightarrow V = \dfrac{\sqrt{38}}{4}$

Correct option is (a)

Example 4: For any vector $\vec{a}$, the value of $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$ is equal to

(a) $|\vec{a}|^2$ (b) $2|\vec{a}|^2$ (c) $\vec{a}^2$ (d) $2\vec{a}^2$

Soln: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ (Here, a_1, a_2, a_3 are not constants)

$\therefore \quad \vec{a} \times i = a_2\hat{k} - a_3\hat{j}$

$\left|\vec{a} \times \hat{i}\right| = \sqrt{a_2^2 + a_3^2}$

Similarly, $\left|\vec{a} \times \hat{j}\right| = \sqrt{a_1^2 + a_3^2}$ and $\left|\vec{a} \times \hat{k}\right| = \sqrt{a_1^2 + a_2^2}$

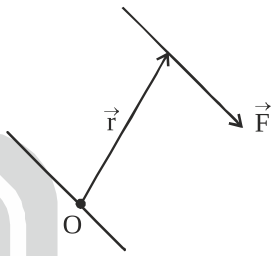
$\therefore \quad \left|\vec{a} \times \hat{i}\right|^2 + \left|\vec{a} \times \hat{j}\right|^2 + \left|\vec{a} \times \hat{k}\right|^2 = 2\left(a_1^2 + a_2^2 + a_3^2\right) = 2|\vec{a}|^2$

Correct option is (b)

Example 5: If $\vec{F} = \hat{i} + 3\hat{j} - \hat{k}$ acts at the point (1, 1, 1), then the torque of $\vec{F}$ about the line $\vec{r} = 3\hat{i} + 2\hat{k} + (2\hat{i} - 2\hat{j} + \hat{k})t$ is_________.

Soln: We first find the vector torque about a point on the line, say the point (3, 0, 2).

This is $\vec{r} \times \vec{F}$, where $\vec{r}$ is the vector from the point about which we want the torque, to the point at which $\vec{F}$ acts, that is, from (3, 0, 2) to (1, 1, 1).



$\therefore \quad \vec{r} = \left(\hat{i} + \hat{j} + \hat{k}\right) - \left(3\hat{i} + 2\hat{k}\right) = -2\hat{i} + \hat{j} - \hat{k}$

$$\therefore \quad \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 2\hat{i} + 3\hat{j} - 7\hat{k}$$

The torque about the line is $\hat{n} \cdot \left(\vec{r} \times \vec{F}\right)$ where $\hat{n}$ is a unit vector along the line $\hat{n} = \frac{1}{3}\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$

Therefore, torque about the line, $\hat{n} \cdot \left(\vec{r} \times \vec{F}\right) = 1$

Correct answer is (1)

■ Scalar Triple Product:

Scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are defined as

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = [abc]$$

Note:

(i) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$ i.e. $[abc] = [bca] = [cab]$

(ii) Volume of a parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as concurrent edges is :$V = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$