generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

8.4 Invariance of the equations of motion:

The equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time i.e., if $\bar{L}(q,\dot{q},t)$ is new lagrangian defined as:

$$\bar{L}(q,\dot{q},t)=L(q,\dot{q},t)+\frac{d}{dt}G(q,t)$$

Then L and $\bar{L}$ give the same equations of motion.

Theorem of quadratic nature of kinetic energy :

The kinetic energy of a system is, in general, a quadratic function of generalised velocities and, in particular, a homogeneous quadratic function of generalised velocities when the kinetic energy is independent of time.

Proof : From the fundamental formula, we have

$$T=\sum_{i=1}^{i=n}\frac{1}{2}m_i\dot{r}_i^2=\sum_{i=1}^{i=n}\frac{1}{2}m_i\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i \quad \left(\because \dot{\vec{r}}_i\cdot\dot{\vec{r}}_i=\dot{r}_i^2\right)$$

where n is the number of particles of the system, m_i is the mass of the ith particle and $\dot{r}_i$ is the velocity of the i^{th} particle.

Now, $\quad \vec{r}_i=\vec{r}_i\left(q_1,q_2,......,q_f,t\right)$ where f is the number of degrees of freedom.

$$\therefore \quad \dot{\vec{r}}_i=\frac{d\vec{r}_i}{dt}=\sum_{k=j}^{k=f}\frac{\partial\vec{r}_i}{\partial q_k}\frac{dq_k}{dt}+\frac{\partial\vec{r}_i}{\partial t}=\sum_{k=1}^{k=f}\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k+\frac{\partial\vec{r}_i}{\partial t}$$

$$\therefore \quad T=\sum_{i=1}^{i=n}\frac{1}{2}m_i\left(\sum_{k=1}^{k=f}\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k+\frac{\partial\vec{r}_i}{\partial t}\right)\cdot\left(\sum_{j=1}^{j=f}\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial\vec{r}_i}{\partial t}\right)$$

$$\text{Or,} \quad T=\sum_{i=1}^{i=n}\frac{1}{2}m_i\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}\left(\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_k\dot{q}_j+\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial t}\dot{q}_k+\frac{\partial\vec{r}_i}{\partial t}\cdot\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial\vec{r}_i}{\partial t}\cdot\frac{\partial\vec{r}_i}{\partial t}\right)$$

Since the second and third terms are identical, they may be added.

$$\therefore \quad T=\frac{1}{2}\sum_{i=1}^{i=n}m_i\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_k\dot{q}_j+\frac{1}{2}\sum_{i=1}^{i=n}m_i\left(\frac{\partial\vec{r}_i}{\partial t}\right)\sum_{k=1}^{k=f}2\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k+\sum_{i=1}^{i=n}\frac{1}{2}m_i\left(\frac{\partial\vec{r}_i}{\partial t}\right)^2$$

$$=\sum_{k=1}^{k=f}\sum_{j=1}^{j=f}a_{k_j}\dot{q}_k\dot{q}_j+\sum_{k=1}^{k=f}a_k\dot{q}_k+a \qquad \text{... (A)}$$

Where, $a_{k_j}=\sum_{i=1}^{i=n}\frac{1}{2}m_i\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial q_j}$, $a_k=\sum_{i=1}^{i=n}m_i\frac{\partial\vec{r}_i}{\partial q_k}\cdot\frac{\partial\vec{r}_i}{\partial t}$

$$\therefore \quad a=\sum_{i=1}^{i=n}\frac{1}{2}m_i\left(\frac{\partial\vec{r}_i}{\partial t}\right)^2$$

From this equation (A), it is seen that the kinetic energy is, in general, a quadratic function of generalised velocities. A case of of considerable importance arises when t is not explicitly involved in the transformation equation. Then $\frac{\partial\vec{r}_i}{\partial t}=0$ and therefore, $a_k=0$ and $a=0$, so the above equation reduces to

$$T = \sum_{k=1}^{k=f} \sum_{j=1}^{j=f} a_{jk} \dot{a}_k \dot{q}_j$$

Thus the kinetic energy is a homogeneous quadratic expression of generalised velocities when the transformation equation is $\vec{r}_i = \vec{r}_i(q_j)$, i.e., it is independent of time. In a free system, conservative system or system with scleronomous constraints the kinetic energies are independent of time and so their kinetic energies are homogeneous quadratic function of generalised velocities.

8.5 Cyclic coordinates and Conservation laws:

Cyclic or ignorable coordinates : If the Lagrangian of a system is not the function of a given coordinate, then that coordinate is said to be cyclic or ignorable coordinate of the system. If the coordinate q_k does not occur in the expression for L, then q_k is said to be ignorable or cyclic.

In such case $\frac{\partial L}{\partial q_k} = 0$

(a) Conservation theorem for generalised momentum : The theorem is : the generalised momentum corresponding to a cyclic or ignorable coordinate of a system is conserved.

Proof : According to Lagrange's equations for a conservative system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

If the k-th generalised coordinate is ignorable, then $\frac{\partial L}{\partial q_k} = 0$

With this substitution in the Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \text{ or } \frac{\partial L}{\partial \dot{q}_k} = a \text{ (where } a \text{ is a constant)}$$

But $\frac{\partial L}{\partial \dot{q}_k} = p_k$, $k\text{-}th$ generalised momentum of the sytem.

Therefore, $p_k = \text{constant}$

(b) Conservation theorem for energy : The principle of homogeneity of time : The theorem states : If the Lagrangian of a system does not contain time explicitly, then the quantity defined by $H = \sum_i p_k \dot{q}_k - L$, known as the Hamiltonian is a constant. If the transformation equation between the generalized coordinates and normal coordinates is independent of time, i.e., $\vec{r} = \vec{r}(q_k)$ and $V = V(q_k)$, then the Hamiltonian is equal to the total energy of the system and it remains conserved. This follows from the principle of homogeneity of time which states that the laws of motion of a system does not depend on the choice of time reference.

Proof : If lagrangian does not contain time, then it is a function of generalised coordinates and velocities only

That is, $L = L(q, \dot{q})$

The total time derivative of it is given by

$$\frac{dL}{dt} = \sum_{k=1}^{k=f}\left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right)$$

where, f = number of degrees of freedom.

Using Lagrange's equations $\frac{d}{dt}\left(\frac{dL}{d\dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$ we have after replacing $\frac{\partial L}{\partial q_k}$ in the summation by $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$

$$\frac{dL}{dt} = \sum_{k=1}^{k=f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right] = \sum_{k=1}^{k=f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right]$$

Using the rule for differentiation of the product $\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k$, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k = \sum_{k=1}^{k=f}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right)$$

Interchanging summation with the total time differential, we have

$$\frac{dL}{dt} = \frac{d}{dt}\sum_{k=1}^{k=f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k$$

Or, $$\frac{d}{dt}\left[L - \sum_{k=1}^{k=f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right] = 0$$

Or, $$L - \sum_{k=1}^{k=f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k = a \text{ constant}$$

$\Rightarrow$ $$H = \text{constant}$$

Now, let V be a function of generalized coordinates alone. Then,

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V) = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} \quad \left(\text{Since } \frac{\partial V}{\partial \dot{q}_k} = 0\right)$$

Also, the kinetic energy T is a homogeneous second degree function of generalised velocities, that is,

$T = \left(a_1\dot{q}_1^2 + a_2\dot{q}_2^2 + ...+ \dot{q}_f^2 a_f + b_1\dot{q}\dot{q}_2 + ...\right)$.

Where a's and b's are constants depending on masses of particles and generalised coordinates.
By Euler's theorem,

$$\sum_{k=1}^{k=f}\dot{q}_k\frac{\partial T}{\partial \dot{q}_k} = 2T \quad \text{or} \quad \sum_{k=1}^{k=f}\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} = 2T$$

$\therefore$ $$L - 2T = a \text{ constant} \quad \text{or} \quad (T - V) - 2T = a \text{ constant}$$

Or, $$T + V = \text{Total energy} = E = \text{constant}$$

(c) Conservation of linear momentum : The principle of homogeneity of space. The theorem states that the linear momentum of a closed system (that is, a system free form external forces) is a constant (called an integral of motion) of the system.

Proof : The law follows from the **principle of homogeneity of space** which means that a parallel translation of a closed system as a whole in no way changes the mechanical property of the system i.e. its Lagrangian function is left unchanged. In other words we may say that the principle of homogeneity of space means that mechanical property of a closed system cannot change if origin of the frame of reference is chanaged.

The Lagrangian function of a closed system consisting of *n* particles is given by

$$L = T - V = \sum_{i=1}^{i=n} \frac{1}{2} m_i \dot{r}_i^2 - V\left(\vec{r}_1 \cdot \vec{r}_2 ......\vec{r}_n\right)$$

The change in the Lagrangian function upon a parallel translation of the system through an infinitesimal distance, specified by an arbitrary value δs, is

$$\partial L = \sum_{i=1}^{i=n} \frac{\partial L}{\partial r_i} \delta s = \delta s \sum_{i=1}^{i=n} \frac{\partial L}{\partial r_i}$$

According to the principle of homogeneity of space this change of the Lagrangian function must be zero. Hence, for compliance with this fundamental principle we have

$$\sum_{i=1}^{i=n} \frac{\partial L}{\partial r_i} = 0$$

According to Lagranges equation of motion, equation for the linear motion of the *i*-th particle is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = Q'_i \text{ or } \sum_{i=1}^{i=n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \sum_{i=1}^{i=n} \frac{\partial L}{\partial r_i} = \sum_{i=1}^{i=n} Q'_i$$

Where Q'_i is the resultant of all non-conservative forces acting on the ith particle of the system. In case of a closed system, these forces are only internal in origin as a closed system means a system free from external forces, that is, for such a system $\sum_{i=1}^{i=n} Q'_i = 0$ because internal forces being equal and opposite their sum is zero.

$$\therefore \quad \sum_{i=1}^{i=n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = 0$$

Or, $$\sum_{i=1}^{i=n} \frac{\partial L}{\partial \dot{r}_i} = a \text{ constant}, \text{ or } \sum_{i=1}^{i=n} m_i v_i = a \text{ constant}$$

$$\left(\because \frac{\partial L}{\partial \dot{r}_i} = \text{linear momentum of } i\text{-th particle}\right)$$

CONSTANT OF THE MOTION

The dynamical state of a particle is completely determined at a given time *t*, if we know its position *x*, and its velocity $\dot{x}$. If we know the dynamical state of a particle at some time t_0, and if the force *f* acting on the particle is a known function of x, $\dot{x}$ and t we can determine the state of the particle at any later time time from the equations of motion

$$f_i = m\ddot{x}_i; \quad i = 1, 2, 3 \qquad ... (1)$$

This set of equations contains of three second order differential equations in the three variables x_1, x_2, and x_3. The solution of this set of equations for x and $\dot{x}$ can be written in the form

$$x_i = x_i(c, t); \quad i = 1, 2, 3 \qquad ... (2)$$

$$\dot{x}_i = \dot{x}_i(c, t); \quad i = 1, 2, 3 \qquad ... (3)$$

where $c \equiv c_1, ..., c_6$ is a set of six independent and arbitrary constants. If we know the values of x and $\dot{x}$ at some instant of time *t*, the values of the six constant will be uniquely determined. Conversely if the six constants are known, the values of x and $\dot{x}$ at any arbitrary time *t* are determined.

The set of six equations given by Eqs. (2) and (3) can be solved for the set of six constants c_i. If we do this we obtain six equations,