In this analysis, scattering amplitude can be written as

$$f(\theta)=\frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{2i\delta_\ell}-1\right]P_\ell(\cos\theta)$$

$$\Rightarrow f(\theta)=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_\ell}\sin\delta_\ell\;P_\ell(\cos\theta)$$

where $k^2=\frac{2\mu E}{\hbar^2}$ and δ_ℓ is the phase shift of the individual partial waves due to scattering.

The corresponding differential cross-section will be

$$\frac{d\sigma}{d\theta}=\left|f(\theta)^2\right|=\frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_\ell}\sin\delta_\ell\;P_\ell(\cos\theta)\right|^2$$

Therefore, the total cross-section will be

$$\sigma_t=\int_0^{\pi}\int_0^{2\pi}\frac{d\sigma}{d\Omega}\sin\theta\,d\theta\,d\phi=2\pi\int_0^{\pi}\left|f(\theta,\phi)\right|^2\sin\theta\,d\theta\qquad=\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

Example: For S waves $(\ell=0)$,

Scattering amplitude $f(\theta)=\frac{1}{k}e^{i\delta_0}\sin\delta_0$

Differential cross section $\frac{d\sigma}{d\Omega}=\left|f(\theta)\right|^2=\frac{1}{k^2}\sin^2\delta_0$

Total cross section $\sigma=\frac{4\pi}{k^2}\sin^2\delta_0$

Note:

(i) Total cross section does not depend upon on the scattering angle θ.

(ii) At sufficiently low energies, s-wave scattering will be dominating one. As energy increases, contributions from the higher value of l values become important.

Optical Theorem

Now, for $\theta=0$

$$f(0)=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_l}\sin\delta_\ell=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\cos\delta_\ell\sin\delta_\ell+i\sin^2\delta_\ell\right]$$

Now, $$\sigma_t=\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell=\frac{4\pi}{k}\operatorname{Im}\left[f(0)\right]$$

This relation is known as Optical theorem.

Phase shift:

$$\sin\delta_\ell=-\frac{2\mu k}{\hbar^2}\int_0^{\infty}V(r)\,j_l^2(kr)\,r^2dr$$

For attractive potential $V(r)=-ve$ $\Rightarrow$ positive phase shift.

For repulsive potential V(r) = + ve $\Rightarrow$ negative phase shift.