3

Limit, Continuity

Definition: Limit of a function $f(x)$ is said to exist, as $x \to a$ a wehen

$$\lim_{h\to 0^+} f(a-h) \quad = \quad \lim_{h\to 0^+} f(a+h) = \text{finite}$$

(Left hand limit) (Right hand limit)

Note that we are not intersted in knowing about what happens at x = a. Also note that if L.H.L & R.H.L are both tending towards $'\infty' or '-\infty'$, then it is said to be infinite limit.

Remember, $x \to a$, means that x is approcing to 'a' not equl to 'a'

Fundamental theorems on limits :

Let $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m.$ if l & m are finite, then

(a) $\lim_{x\to a}\{f(x) \pm g(x)\} = \ell \pm m.$

(b) $\lim_{x\to a}\{f(x)\cdot g(x)\} = \ell \cdot m.$

(c) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\ell}{m},$ provided $m \neq 0$

(d) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = k\ell;$ where k is a constant

(e) $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(m);$ provided f is continuous at $g(x) = m.$

Standard Limits:

(i) $\lim_{x\to 0}\dfrac{\sin x}{x} = \lim_{x\to 0}\dfrac{\tan x}{x} = 1$

(ii) $\lim_{x\to 0}\dfrac{\tan^{-1} x}{x} = \lim_{x\to 0}\dfrac{\sin^{-1} x}{x} = 1$

(iii) $\lim_{x\to 0}(1+x)^{1/x} = e;\quad \lim_{x\to 0}(1+ax)^{1/x} = e^a$

(iv) $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e;\quad \lim_{x\to\infty}\left(1+\dfrac{a}{x}\right)^x = e^a$

(v) $\lim_{x\to 0}\dfrac{e^x-1}{x} = 1;\quad \lim_{x\to 0}\dfrac{a^x-1}{x} = \log_e a = \ell na \;, a > 0$

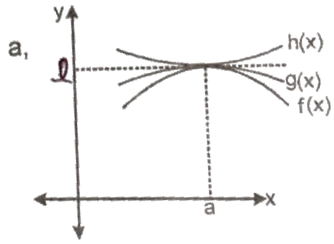
(vi) $\lim_{x\to 0}\dfrac{\ell n(1+x)}{x} = 1$

(vii) $\lim_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$

Sandwitch theorem or squeeze paly theorem

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at $x = a$ itself.

suppose also that



$$\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x),$$

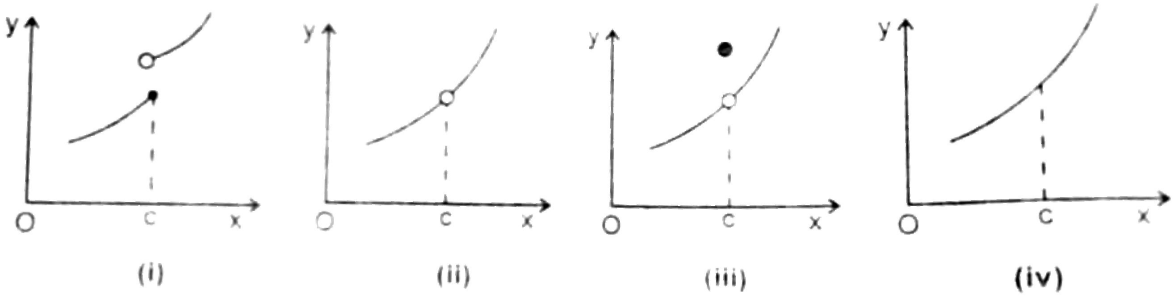
then $\lim_{x\to a} g(x) = \ell$.

Contiinuity & derivability:

A function $f(x)$ is said to be continuous at $x = c$, if $\lim_{x\to c} f(x) = f(c)$

i.e, $\lim_{h\to 0^+} f(c-h) = \lim_{h\to 0^+} f(c+h) = f(c)$

If a function $f(x)$ is continuous at $x = c$, the graph of $f(x)$ at the corresponding point $(c, f(c))$ will not be broken. But if $f(x)$ is discontinuous at x = c, then graph will be broken when $x = c$



((i), (ii) and (iii) are discontinuous at $x = c$)
((iv) is continuosu at $x = c$)
A function f can be discontinuous due to any of the following three reasons:

(i) $\lim_{x\to c} f(x)$ does not exist i.e., $\lim_{x\to c^-} f(x) \ne \lim_{x\to c^+} f(x)$ (Fig.i)

(ii) f(x) is not defined at x =c (Fig.ii)

(iii) $\lim_{x\to c} f(x) \ne f(c)$, geometrically, the graph of the function will exhibit a break at $x = c$ (Fig.iii)

Differentiablility of a Function at a Point:

(i) The right hand derivative of $f(x)$ at $x = a$ denoted by $f'(a^+)$ is defined by:

R.H.D. = $f'(a^+) = \lim_{h\to 0^+} \dfrac{f(a+h) - f(a)}{h}$, provided the limit exists.

(ii) The left hand derivative of $f(x)$ at $x = a$ denoted by $f'(a^-)$ is defined by:

L.H.D. = $f'(a^-) = \lim_{h\to 0^+} \dfrac{f(a+h) - f(a)}{-h}$, provided the limit exists.

A function $f(x)$ is said to be differentiable at $x = a$ *if* $f'(a^+) = f'(a^-) =$ finite

By definition $f'(a) = \lim_{h\to 0^+} \dfrac{f(a+h) - f(a)}{h}$

Ex. The function $f(x)=\begin{cases} ax^2+b & \text{if } x\le 2 \\ x^3 & \text{if } x>2 \end{cases}$ is both continuous and differentiable

(a) When $a = 1$ and $b = 2$
(b) When $a = 3$ and $b = 1$
(c) For all values of a and b satisfying the equation $4a + b = 8$
(d) When $a = 3$ and $b = -4$

Soln. f is continuous at $x = 2$

$$\Rightarrow \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(ax^2+b)=\lim_{x\to 2^+} x^3=4a+b$$

$$\Rightarrow 4a+b=8=4a+b$$

$$\Rightarrow 4a+b=8 \qquad ....(i)$$

Also, f is differentiable at $x = 2$

$\Rightarrow$ LHD at $x = 2$ = RHD at $x = 2$

$$\Rightarrow \left[\frac{d}{dx}(ax^2+b)\right]_{x=2}=\frac{d}{dx}\left[x^3\right]_{x=2}$$

$$\Rightarrow \left[2ax\right]_{x=2}=\left[3x^2\right]_{x=2}$$

$$\Rightarrow 4a=12$$

$$\Rightarrow a=3$$

from (i), $b = 8 - 4a = 8 - 4 \times 3 = 8 - 12 = -4$

$b = -4$

Correct option is (d)

Ex. The function $\sin|x|$ is

(a) Continuous and differentiable at $x=\pi$
(b) Continuous but not differentiable at $x=\pi$
(c) Differentiable but not continuous at $x=\pi$
(d) Neither continuous nor differentiable at $x=\pi$

Soln. Let $f(x) = \sin|x|$

$$\lim_{x\to\pi} f(x)=\lim_{x\to\pi}\left(\sin|x|\right)=\sin(\pi)=0$$

$$=\lim_{x\to\pi} f(\pi)=f(\pi)$$

$\Rightarrow f$ is continuous at $x=\pi$

Again, $$f'(\pi)=\lim_{x\to\pi}\frac{f(x)-f(\pi)}{x-\pi}$$

$$=\lim_{x\to\pi}\frac{\sin|x|-\sin(\pi)}{x-\pi}=\lim_{x\to\pi}\frac{\sin|x|}{x-\pi}=\lim_{x\to\pi}\frac{\sin(x)}{x-\pi}$$

$$=\lim_{x\to\pi}\frac{\cos(x)}{1} \qquad \text{(L'Hospital rule)}$$

$$=\cos(\pi)=-1$$

Thus, f is differentiable at $x=\pi$ & $f'(\pi)=-1$

Correct option is (a)

Differentiation:

Derivative of a function : Derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ and defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Provided that limit exists

Algebra of derivative of function :

(1) **Sum rule** $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$

(2) **Difference rule** $\frac{d}{dx}\left(f(x)-g(x)\right) = \frac{d}{dx}\left(f(x)\right) - \frac{d}{dx}\left(g(x)\right)$

(3) **Product rule** $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = g(x)\cdot\frac{d}{dx}\left(f(x)\right) + f(x)\cdot\frac{d}{dx}\left(g(x)\right)$

(4) **Quotient rule** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\frac{d}{dx}\left(f(x)\right) - f(x)\cdot\frac{d}{dx}\left(g(x)\right)}{\left(g(x)\right)^2}$

(5) **Scalar multiplication rule** $\frac{d}{dx}\left(\alpha.f(x)\right) = \alpha.\frac{d}{dx}\left(f(x)\right)$

(6) **Chain Rule** $\frac{d}{dx}\left(f(g(x)\right) = f'\left(g(x)\right).g'(x)$

Derivative of special functions

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$$

$$\frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}(\tan x) = \sec^2(x)$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2(x)$$

$$\frac{d}{dx}(\sec x) = \sec(x)\cdot\tan(x)$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec}(x)\cdot\cot(x)$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos ec^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{-n}{x^{n+1}}$$

$$\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

$$\frac{d}{dx}\left(\log x\right)=\frac{1}{x}$$

$$\frac{d}{dx}(a^x)=a^x\log(a)$$

$$\frac{d}{dx}(e^x)=e^x$$

Logarthmic Differentiation

If $f(x)=(u(x))^{v(x)}$

$\Rightarrow \log(f(x))=v(x).\log(u(x))$

Differentiating both side w.r.t.x

$$\Rightarrow \frac{d}{dx}\left[\log(f(x))\right]=\frac{d}{dx}\left[v(x)\cdot\log(u(x))\right]$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x)=v(x)\cdot\frac{1}{u(x)}\cdot u'(x)+v'(x)\log(u(x))$$

$$\Rightarrow f'(x)=f(x)\left[\frac{v(x)u'(x)}{u(x)}+v'(x).\log(u(x))\right]$$

Exponential Differentiation

If $f(x)=a^{g(x)}$ where $a>0, a\neq 1$ then $\frac{d}{dx}\left(f(x)\right)=\frac{d}{dx}\left(a^{g(x)}\right)=a^{g(x)}.g^1.(x)\log(a)$

Parametric Differentiation

If $y=f(t)$ and $x=g(t)$ where t is parameter

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}$$

Ex. If $y = x^y$ then $\frac{dy}{dx}$ is

(a) $\frac{y^2}{x(1-y\log(x))}$ (b) yx^{y-1}

(c) $\frac{y}{x(1-y\log(x))}$ (d) $yx^{y-1}\log(x)$

Soln. $y = x^y$

taking log both side

$\Rightarrow \log(y) = \log(x^y)$

$\Rightarrow \log(y) = y\log(x)$

Differentiating both side w.r.t. x

$$\Rightarrow \frac{d}{dx}(\log(y)) = \frac{d}{dx}(y\log(x))$$

$$\Rightarrow \frac{1}{y}\times\frac{dy}{dx} = y\cdot\frac{1}{x} + \log(x)\cdot\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{1-y\log(x)}{y}\cdot\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1-y\log(x))}$$

Correct option is (a)

Ex. The derivative of $x\log(x)$ with respect to x is

(a) $\log(ex)$ (b) $x + \log(x)$ (c) $1 + \frac{1}{x}$ (d) $x + \frac{1}{x}$

Soln. $\frac{d}{dx}(x\log(x))$

$$= x\cdot\frac{1}{x} + 1.\log(x) = 1 + \log(x) = \log(e) + \log(x)$$

$= \log(ex)$

Correct option is (a)

Ex. The differential Co-efficient of $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ is

(a) $\frac{-4x}{(1+x)^2}$ (b) $\frac{-8x}{(1+x)^3}$ (c) $\frac{-8x^2}{(1+x)^3}$ (d) None

Soln. Differential Co-efficient $= \frac{d}{dx}\left(\sin^{-1}\left(\frac{1-x}{1+x}\right)\right)$

$$= \frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}} \times \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1+x}{\sqrt{(1+x)^2-(1-x)^2}} \times \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$= \frac{1+x}{\sqrt{2.(2x)}} \times \frac{-1-x-1+x}{(1+x)^2} = \frac{1+x}{2\sqrt{x}} \cdot \frac{-2}{(1+x)^2} = \frac{-1}{(1+x)\sqrt{x}}$$

Correct option is (d)

Ex. If $f(x+y) = f(x)\cdot f(y)$ for all x and y and $f(5)=2,\ f'(0)=3$ then $f'(5)$ is

(a) 5 (b) 6 (c) 0 (d) 3

Soln. $f'(5) = \lim_{x\to 5} \frac{f(x)-f(5)}{x-5} = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$

$$= \lim_{h\to 0} \frac{f(h)\cdot f(5)-f(5)}{h} = \lim_{h\to 0} \frac{2\cdot\left(f(h)-1\right)}{h}$$

Since $f(x+y) = f(x)\cdot f(y)$

$\Rightarrow\ f(1+0) = f(1)\cdot f(0) \Rightarrow f(1) = f(1)\cdot f(0)$

$\Rightarrow\ f(0) = 1$

So, $f'(5) = \lim_{h\to 0} \frac{2\left(f(h)-f(0)\right)}{h-0} = 2\cdot f'(0) = 2.3 = 6$

Correct option is (b)

Ex. If $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a) f is not differentiable at 0 (b) $f'(0) = 0$

(c) $f'(0) = 1$ (d) $f'(0) = 2$

Soln. $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)-0}{x}$

$= \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

f is differentiable & $f'(0) = 0$

Correct option is (b)

Ex. The function $f(x) = x|x|$ is

(a) differentiable at $x = 0$ and $f'(0) = 0$
(b) not differentiable through continuous at $x = 0$
(c) not continuous at $x = 0$
(d) differentiable at $x = 0$ and $f'(0) = 1$

Soln. $f(x) = x|x| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

$$LHD = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{-x^2 - 0}{x}$$

$$= \lim_{x\to 0^-}(-x) = 0$$

$$RHD = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^2 - 0}{x - 0} = \lim_{x\to 0}(x) = 0$$

Since LHD = RHD at $x = 0$

Therefore, f is differentiable at $x = 0$

$\Rightarrow$ f is continuous at $x = 0$

Also, $f'(0) = 0$

Correct option is (a)

Ex. If $y = x^x$, then $\frac{dy}{dx}$ is equal to

(a) $x^x(x - \ln(x))$ (b) $x^x(1 - \ln(x))$ (c) $x^x(1 + \ln(x))$ (d) $x^x(x + \ln(x))$

Soln. $y = x^x$

$\Rightarrow \log(y) = x\log(x)$

Differentiating both side w.r.t. x

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(x\log(x))$$

$$\Rightarrow \frac{1}{y}\times\frac{dy}{dx} = x\cdot\frac{1}{x} + 1.\log(x))$$

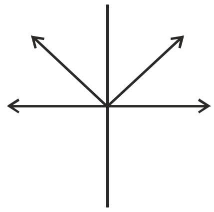
$$\Rightarrow \frac{dy}{dx} = y(1 + \log(x)) = x^x(1 + \log(x))$$

Correct option is (c)

Ex. The function $f(x) = |x|$ is

(a) Continuous but not differentiable in $\mathbb{R}$
(b) Continuous and differentiable in $\mathbb{R}$
(c) Not continuous but differentiable in $\mathbb{R}$
(d) Neither continuous nor differentiable in $\mathbb{R}$

Soln. $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$



$\lim\limits_{x\to 0^-} |x| = \lim\limits_{x\to 0^+} |x| = f(0) = 0$

$\therefore$ f is continuous at $x = 0$

$\Rightarrow$ f is continuous on $\mathbb{R}$

But LHD $= \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{-x-0}{x} = -1$

RHD $= \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \dfrac{x-0}{x} = 1$

since LHD $\neq$ RHD

Therefore, f is not differentiable at $x = 0$

Hence, $f(x) = |x|$ is contiuous but not differentiable in $\mathbb{R}$.

Correct option is (a)

Ex. Differentiate ln(x) with respect to $\dfrac{1}{x}$

(a) $\dfrac{-1}{x}$ (b) $\dfrac{1}{x}$ (c) x (d) $-x$

Soln. Let $u = \ln(x), v = \dfrac{1}{x}$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -x$$

Correct option is (d)

Ex. The derivative of $\cos^{-1}(\sin(x))$ with respect to x is

(a) 0 (b) cot$x$ (c) –1 (d) $\sin(x)\cdot\cos(x)$

Soln. $\dfrac{d}{dx}\left(\cos^{-1}(\sin(x))\right)$

$$= \frac{d}{dx}\left(\cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)\right) = \frac{d}{dx}\left(\frac{\pi}{2}-x\right) = -1$$

Correct option is (c)

Ex. If $x^y = e^{x-y}$ then $\dfrac{dy}{dx}$ is

(a) $\dfrac{\log(x)}{[\log(ex)]^2}$. (b) $\dfrac{y\log(x)}{x[\log(ex)]}$ (c) $\dfrac{x\log(x)}{[\log(e^x)]^2}$ (d) $\dfrac{x\log(x)}{[\log(e^x)]}$

Soln. $x^y = e^{x-y}$

taking log both side

$\Rightarrow\ y\log(x) = (x-y)\log(e)$

$\Rightarrow\ y\log(x) = x-y$ .....(i)

Differentiating both side w.r.t.x.

$$\Rightarrow\ \frac{d}{dx}(y\log(x)) = \frac{d}{dx}(x-y)$$

$$\Rightarrow\ y\cdot\frac{1}{x} + \frac{dy}{dx}\cdot\log(x) = 1 - \frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}(1+\log(x)) = 1 - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x-y}{x(1+\log x)} \qquad (\text{ from (i) and } \log e = 1)$$

$$\frac{dy}{dx} = \frac{y\log x}{x\log(ex)} \qquad (\because x-y = y\log x)$$

Correct option is (b)

Ex. If y is a function of x defined

$$y = a^x + \cfrac{1}{a^x + \cfrac{1}{a^x + \cfrac{1}{a^x + ...}}}$$

The derivative of y with respect to x is

(a) $\dfrac{ya^x\ln(a)}{2y+a^x}$ (b) $\dfrac{ya^x\ln(a)}{2y-a^x}$ (c) $\dfrac{ya^x\ln(a)}{2(y-a^x)}$ (d) $\dfrac{ya^x\ln(a)}{2(y+a^x)}$

Soln. $y = a^x + \dfrac{1}{y}$

$\Rightarrow\ y - \dfrac{1}{y} = a^x$ ....(i)

$$\Rightarrow\ \frac{dy}{dx} + \frac{1}{y^2}\cdot\frac{dy}{dx} = a^x\log(a)$$

$$\Rightarrow\ \frac{dy}{dx}\left(1+\frac{1}{y^2}\right) = a^x\log(a)$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{y^2a^x\log(a)}{y^2+1} = \frac{ya^x\log(a)}{y+\frac{1}{y}}$$

from (i),

$$y-\frac{1}{y}=a^x$$

$$\Rightarrow \left(y+\frac{1}{y}\right)-\frac{2}{y}=a^x$$

$$\Rightarrow \left(y+\frac{1}{y}\right)-2\left(\frac{1}{y}+y\right)=a^x-2y$$

$$\Rightarrow -\left(y+\frac{1}{y}\right)=a^x-2y$$

$$\Rightarrow y+\frac{1}{y}=2y-a^x$$

So, $\frac{dy}{dx}=\frac{ya^x\log(a)}{2y-a^x}$

Correct option is (b)

Ex. The value of the derivative of $y=\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right], x\neq 0$ at $x=1$ is

(a) $\frac{1}{4-2\sqrt{2}}$ (b) $\frac{1}{4+2\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Soln. $y=\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Let $x=\tan\theta$

$$y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$y=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) \qquad \left(\because \sec\theta=\frac{1}{\cos\theta},\ \tan\theta=\frac{\sin\theta}{\cos\theta}\right)$$

$$y=\tan^{-1}\left(\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right)$$

$$y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}(x)$$

$$y = \frac{1}{2}\tan^{-1}(x) \Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2(1+1)} = \frac{1}{4}$$

Correct option is (d)

Ex. If y is a function of x given by $y = \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+.........}}}}$, then $\frac{dy}{dx}$ at $(0, 0)$ is

(a) –1 (b) 0 (c) 1 (d) $\sqrt{2}$

Soln. $y = \sqrt{x+y}$

$\Rightarrow\ y^2 = x + y$

Differentiating with respect to x

$$\Rightarrow\ 2y.\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow\ (2y-1)\frac{dy}{dx} = 1$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{2y-1}$$

At (0,0) $\frac{dy}{dx} = \frac{1}{2\times 0 - 1} = -1$

Correct option is (a)

Ex. The value of $\frac{d}{dx}\left(x\ln(x) + (1-x)\ln(1-x)\right)$ is

(a) $\ln\left(\frac{x}{1-x}\right)$ (b) $\ln\left(\frac{1-x}{x}\right)$ (c) 0 (d) $2+\ln\left(\frac{x}{1-x}\right)$

Soln. $\frac{d}{dx}\left(x\ln(x) + (1-x)\ln(1-x)\right)$

$$= \frac{d}{dx}[x\ln(x)] + \frac{d}{dx}\left[(1-x)\ln(1-x)\right]$$

$$= x.\frac{1}{x} + 1.\ln(x) + (1-x)\times\frac{1}{1-x}\times(-1) + (-1)\ln(1-x)$$

$$= 1 + \ln(x) - 1 - \ln(1-x)$$

$$= \ln(x) - \ln(1-x)$$

$$= \ln\left(\frac{x}{1-x}\right)$$

Correct option is (a)

Ex. If $u(x)$ and $v(x)$ are differentiable at $x = 0$, and if $u(0) = 5$, $u'(0) = -3$, $v(0) = -1$ and $v'(0) = 2$ then value of $\frac{d}{dx}\left(uv + \frac{u}{v}\right)$ at $x = 0$ is

(a) –20 (b) –7 (c) 6 (d) 3

Soln. $\frac{d}{dx}\left(uv + \frac{u}{v}\right) = \frac{d}{dx}(uv) + \frac{d}{dx}\left(\frac{u}{v}\right)$

$$= u.\frac{dv}{dx} + v\frac{du}{dx} + \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

At $x = 0$,

$$\frac{d}{dx}\left(uv + \frac{u}{v}\right) = u(0).v'(0) + v(0)u'(0) + \frac{v(0).u'(0) - u(0)v'(0)}{(v(0))^2}$$

$$= 5 \times 2 + (-1)\cdot(-3) + \frac{(-1).(-3) - 5.2}{(-1)^2}$$

$$= 10 + 3 + \frac{3-10}{1} = 13 - 7 = 6$$

Correct option is (c)

Ex. If $x^2 + y^2 = \cos(x+y)$ then

(a) $\frac{dy}{dx} = \frac{-(2x + \sin(x+y))}{2y + \sin(x+y)}$ (b) $\frac{dy}{dx} = \frac{2x + \sin(x+y)}{2y + \sin(x+y)}$

(c) $\frac{dy}{dx} = \frac{2x - \sin(x+y)}{2y + \sin(x+y)}$ (d) $\frac{dy}{dx} = \frac{2x - \sin(x+y)}{2y - \sin(x+y)}$

Soln. $x^2 + y^2 = \cos(x+y)$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = -\sin(x+y)\cdot\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = -\sin(x+y) - \sin(x+y)\frac{dy}{dx}$$

$$\Rightarrow 2x + \sin(x+y) = -[\sin(x+y) + 2y]\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x + \sin(x+y)}{2y + \sin(x+y)}$$

Correct option is (a)

Ex. If $2^y = e^x$ then $\frac{dy}{dx}$ is

(a) $ln(2)$ (b) $\frac{1}{ln(2)}$ (c) e^2 (d) $\frac{1}{e^2}$

Soln. $2^y = e^x$

$\Rightarrow \frac{d}{dx}(2^y) = \frac{d}{dx}(e^x)$

$\Rightarrow 2^y ln(2)\frac{dy}{dx} = e^x$

$\Rightarrow \frac{dy}{dx} = \frac{e^x}{2^y ln(2)} = \frac{2^y}{2^y ln(2)} = \frac{1}{ln(2)}$

Correct option is (b)

Ex. If $f(x) = 4x^2 \sin(2x)$ then its first derivative is

(a) $4x^2\cos(2x) + 8x\sin(2x)$ (b) $-4x^2\cos(2x) + 8x\sin(2x)$

(c) $8x^2\cos(2x) + 8x\sin(2x)$ (d) $-8x^2\cos(2x) + 8x\sin(2x)$

Soln. $f(x) = 4x^2\sin(2x)$

$f'(x) = 8x\sin(2x) + 4x^2.2\cos(2x)$

$\Rightarrow f'(x) = 8x\sin(2x) + 8x^2\cos(2x)$

Correct option is (c)

Ex. The derivative of 2^x with respect to x is

(a) $ln(x)\cdot 2^x$ (b) $x\cdot 2^{x-1}$ (c) $l(2)\cdot 2^x$ (d) 2.2^{x-1}

Soln. $\frac{d}{dx}(2^x) = 2^x \ell n(2)$

because $\frac{d}{dx}(a^x) = a^x \ell n(a) \qquad \forall a > 0, a \neq 1$

Correct option is (c)

Second order derivative : If $y = f(x)$ is a function then $\frac{dy}{dx} = f'(x)$ is called first order derivative and

$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is called second order derivative of y *w.r.t.x.*

Ex. If $y = ae^x + be^{-x}$ then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{1}{y}$ (b) y (c) $\frac{dy}{dx}$ (d) y^2

Soln. $y = ae^x + be^{-x}$

$\Rightarrow \frac{dy}{dx} = ae^x - be^{-x}$

$\Rightarrow \frac{d^2y}{dx^2} = ae^x + be^{-x} = y$

Correct option is (b)

5. For large values of n, the value of $\frac{n^2 - n}{n+1}$ tends to **JNU-11**

(a) ∞ (b) 0 (c) 1 (d) an unknown value

Soln. $\therefore$ as $n \to \infty$ we have,

$$\lim_{n\to\infty}\frac{n^2-n}{n+1}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n(n-1)}{n\left(1+\frac{1}{n}\right)}=\infty$$

Correct option is (a)

Ex. The function $f(x)=x|x|$ is

(a) Differentiable at $x=0$ and $\frac{df}{dx}$ at $x = 0$ is zero

(b) Not differentiable through continuous at $x = 0$

(c) Not continuous at $x = 0$

(d) Differentiable at $x = 0$ and $\left.\frac{df}{dx}\right|_{x=0}$ is 1.

Soln. $f(x)=x|x|=\begin{cases} x^2, & x\ge 0 \\ -x^2, & x<0 \end{cases}$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}$$

$$\text{LHD}=\lim_{x\to 0^-}\frac{f(x)}{x}=\lim_{x\to 0^-}\frac{-x^2}{x}=\lim_{x\to 0^-}(-x)=0$$

$$\text{RHD}=\lim_{x\to 0^+}\frac{f(x)}{x}=\lim_{x\to 0^+}\frac{x^2}{x}=\lim_{x\to 0^+}x=0$$

f is differenatiable $x = 0$ and $f'(0)=0$

Correct option is (a)

Ex. Consider the curve $y=\cos(x)-1\ x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$. Tangent at point p on the curve is parallel to the x-axis. Then co-ordinates of the point p are

(a) $(\pi,2)$ (b) $\left(\frac{\pi}{2},-1\right)$ (c) $(\pi,-2)$ (d) $\left(\frac{\pi}{2},1\right)$

Soln. $y=\cos(x)-1$

$$\frac{dy}{dx}=-\sin(x)$$

Since tangent is parallel to x- axis

$$\Rightarrow \frac{dy}{dx}=0\Rightarrow \sin(x)=0\Rightarrow x=n\pi\ \forall\ x\in\mathbb{Z}$$

$$x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\Rightarrow x=\pi$$

$$\Rightarrow y=\cos(\pi)-1=-1-1=-2$$

point p is $(\pi,-2)$

Option (c) is correct