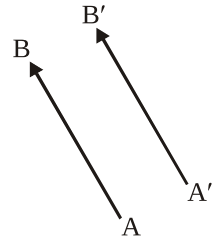
# VECTORS AND SCALARS

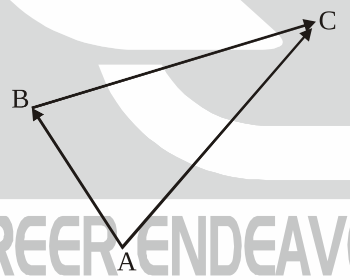
## ★ INTRODUCTION :

A change of position of a particle is called displacement. If a particle moves from a position A to a position B, as shown in the figure.




We can represent its displacement by drawing a line from A to B. The direction of displacement can be shown by putting an arrowhead at B indicating that the displacement was from A to B, the path of the particle need not necessarily be a straight line from A to B; the arrow represents only the net effect of the motion, not the actual motion. Further more, a displacement such $A'B'$, which is parallel to AB, represents the same change in position as AB. We make no distinction between these two displacements. A displacement is therefore, characterised by a length and a direction.

Similarly, by referring to the figure below :

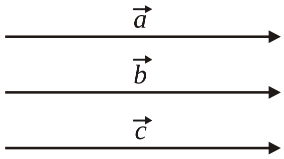



The net effect of the two displacements i.e., A to B and B to C is the same as a displacement from A to C. Therefore, we speak of AC as the sum or resultant of the displacements AB and BC. Notice that this sum is not an algebraic sum and that a number alone cannot uniquely specify it.

Quantities that behave like displacements are called vectors. ***The word 'vector' comes from Latin and means 'carrier'. Vectors then, are quantities that have both magnitude and direction and combine according to certain rules of addition.*** The displacement vector can be considered as the prototype. Some other physical quantities which are vectors, are force, velocity, acceleration, electric field strength and magnetic induction. Many of the laws of physics can be expressed in a compact form using vectors; derivations involving these laws are often greatly simplified if this is done.

***Quantities that can be completely specified by a number and unit and therefore, have magnitude only, are called scalars***. Some scalar quantities are mass, length, time, density, energy and temperature. Scalars can be manipulated by the rules of algebra.

**Equality of Vectors :** Two vectors are said to be equal if they have the same magnitude and direction. In the figure below, three equal vectors $\vec{a}, \vec{b}, \vec{c}$ have been represented. The equality of vectors is expressed in the following way $\vec{a} = \vec{b} = \vec{c}$.




**Equal Vectors**

Since, the three vectors have the same direction, $\hat{a} = \hat{b} = \hat{c}$.
Moreover, the three vectors are also equal in magnitude.

Therefore, $|\vec{a}| = |\vec{b}| = |\vec{c}|$

If the scales selected for the representation of three vectors are the same, then, three equal vectors are represented by three arrows of equal lengths, pointing in the same direction. If, however, the scales chosen are different, then, it is possible to represent equal vectors by arrows of different lengths. Equality of two vectors does not depend upon their location in space. A direct consequence of this physical fact is that a vector can be displaced parallel to itself. This fact may be used in testing the equality of two vectors. It would be of interest to note that, to compare two vectors with different physical dimensions, would be absolutely meaningless. For example, it is not possible to compare a velocity vector with a force vector.

**Multiplication of Vectors by Real Numbers, i.e., Scalar Multiple of a Vector :**

The multiplication of a vector by a real number assumes a lot of significance in such statements as - velocity of car B is double the velocity of car A.
When a vector $\vec{a}$ is multiplied by a real number, say $\lambda$, then we get another vector $\lambda$. The magnitude of $\lambda$ is $\lambda$ times the magnitude of . If $\lambda$ is positive, then the direction of $\lambda$ is the same as that of . If $\lambda$ is negative, then the direction of $\lambda$ is opposite to that of .

If $\vec{a}$ is multiplied by zero, we get a vector whose magnitude is zero and whose direction is arbitrary. This vector is called a zero vector or null vector.

If $\lambda$ is a pure number and has no units, then the units of $\lambda\vec{a}$ are the same as those of . But, if the scalar has a certain unit, then the unit of $\lambda$ will be different from that of $\vec{a}$.

**Example**

- The multiplication of velocity vector by time (a scalar) gives us displacement.
- The multiplication of velocity vector by mass (a scalar) gives us momentum.



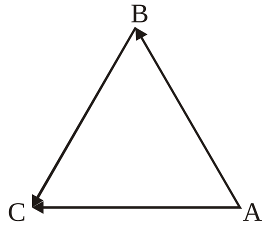
**Addition or Composition of Vectors :** Scalars can be added algebraically. However, vectors do not obey the ordinary laws of algebra. This is because vectors possess both magnitude and direction. Vectors are added geometrically. The process of adding two or more vectors is known as addition or composition of vectors. When two or more vectors are added, the result is a single vector called the resultant vector.

The resultant of two or more vectors is that single vector which alone produces the same effect as that produced by the two individual vectors.

Three laws have been evolved for the addition of vectors :

- Triangle law of vectors for addition of two vectors.
- Parallelogram law of vectors for addition of two vectors.
- Polygon law of vectors for addition of more than two vectors.

**Triangle law of vectors :** Let a particle be at the points A, B, C at three successive times $t$, $t'$ and $t''$ respectively. $\overrightarrow{AB}$ is the displacement vector from time $t$ to $t'$. $\overrightarrow{BC}$ is the displacement vector from time $t'$ to $t''$. The total displacement vector $\overrightarrow{AC}$ is the sum or the resultant of individual displacement vector $\overrightarrow{AB}$ and $\overrightarrow{BC}$.



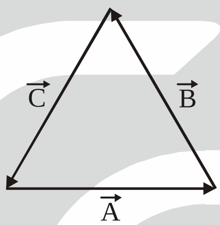

**Triangle law of vectors**

Therefore, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

This leads to the statement of the law of triangle of vectors.

If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then the resultant is represented completely, both in magnitude and direction, by the third side of the triangle taken in the opposite order.

**Corollary :** It follows from triangle law of vectors that, if three vectors are represented by the three sides of a triangle taken in order, then their resultant is zero. Thus, if three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ can be represented completely by the three sides of a triangle taken in order, then their vector sum is zero.



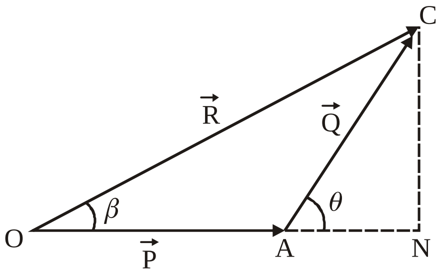

Therefore, $\vec{A} + \vec{B} + \vec{C} = \vec{0}$.

Note that the resultant $\left(-\vec{C}\right)$ of $\vec{A}$ and $\vec{B}$ cancels the third vector $\vec{C}$.

***If the resultant of three vectors is zero, then these can be represented completely by the three sides of a triangle taken in order.***

**To determine analytically the resultant of two vectors, using triangle law of vectors :**

Let two vectors $\vec{P}$ and $\vec{Q}$ acting simultaneously on a particle be represented both in magnitude and direction, by the sides OA and AC of a triangle OAC, taken in order, as shown in the figure. Applying triangle law of vectors, we find that the resultant $\vec{R}$ of the given vectors can be represented by the side OC of the triangle.




In the right angled $\Delta$ONC, considering the magnitudes only.

Equation, $R^2 = ON^2 + NC^2 = (OA + AN)^2 + NC^2$

$$R^2 = (P + AN)^2 + NC^2 \qquad \text{... (1)}$$

In the right angled $\Delta ANC$,

$$\sin\theta = \frac{NC}{Q} \text{ or } NC = Q\sin\theta \text{ and } \cos\theta = \frac{AN}{Q} \text{ or } AN = Q\cos\theta$$

Substituting for NC and AN in equation (1), we have

$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$R^2 = P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$= P^2 + Q^2(\sin^2\theta + \cos^2\theta) + 2PQ\cos\theta$$

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

Therefore, $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$ ... (2)

The above equation (2) gives the magnitude of the resultant vector.

The same equation (2) can also be expressed in the following ways.

$$\left|R^2\right| = \sqrt{\left|\vec{P}\right|^2 + \left|\vec{Q}\right|^2 + 2\left|\vec{P}\right|\left|\vec{Q}\right|\cos\theta}$$

$$\left|\vec{P} + \vec{Q}\right| = \sqrt{\left|\vec{P}\right|^2 + \left|\vec{Q}\right|^2 + 2\left|\vec{P}\right|\left|\vec{Q}\right|\cos\theta}$$

The direction of the resultant can be determined by calculation of $\beta$, which is the angle that $\vec{R}$ makes with $\vec{P}$. Therefore, from the right-angled $\Delta ONC$.

$$\tan\beta = \frac{CN}{ON} = \frac{CN}{(OA + AN)}$$

$$\tan\beta = \frac{Q\sin\theta}{(P + Q\cos\theta)}$$

$$\beta = \tan^{-1}\left[\frac{Q\sin\theta}{(P + Q\cos\theta)}\right]$$

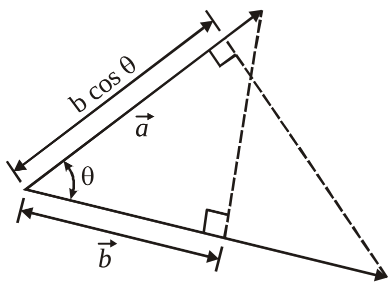
$$\beta = \tan^{-1}\left[\left|\vec{Q}\right|\sin\theta \Big/ \left(\left|\vec{P}\right| + \left|\vec{Q}\right|\cos\theta\right)\right]$$

**<u>Dot Product of Vectors</u> :** The scalar product or product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\cdot\vec{b}$ is defined to be $\vec{a}\cdot\vec{b} = ab\cos\theta$ ... (1)

where $a$ is the magnitude of $\vec{a}$, $b$ is the magnitude of $\vec{b}$ and $\cos\theta$ is the cosine of the angle $\theta$ between the two vectors.

It is important to remember that there are two different angles between a pair of vectors, depending on the direction of rotation. However, only the smaller of the two is taken in vector multiplication.

Since, $a$ and $b$ are scalars and $\cos\theta$ is a pure number, the scalar product of two vectors is a scalar.




**Scalar or dot product of two vector**

As shown in the figure, the scalar product of two vectors can be regarded as the product of the magnitude of one vector and the component of the other vector in the direction of the first.

Therefore, $\vec{a}\cdot\vec{b} = a(b\cos\theta)$ or $b(a\cos\theta)$

Moreover, the scalar product of a vector $\vec{a}$ with the unit vector is the component of $a$ in the direction of the unit vector.

For example : $\vec{A}\cdot\hat{i} = A\times 1\times\cos\theta = A\cos\theta = A_x$

Similarly, $\vec{A}\cdot\hat{j} = A_y$

Therefore, $\vec{A} = \left(\vec{A}\cdot\hat{i}\right)\hat{i} + \left(\vec{A}\cdot\hat{j}\right)\hat{j}$

**Special Cases :**

(1) When both the cars are moving in the same direction, i.e., $\theta = 0$ degrees.

Therefore, $V_{AB} = V_{BA} = \sqrt{V_A^2 + V_B^2 - 2V_AV_B\cos 0^o}$

$\cos 0^o = 1$

Therefore, $V_{AB} = V_{BA} = \sqrt{(V_A - V_B)^2} = \pm(V_A - V_B)$

i.e., $V_{AB} = V_{BA} = (V_A - V_B)$ or $(V_B - V_A)$

Thus, the relative speed between two bodies moving in the same direction is equal to the difference of the individual speeds of two bodies.

(2) When the two cars are moving along parallel lines in opposite direction, i.e., $\theta = 1800$ degrees.

Then, $V_{AB} = V_{BA} = \sqrt{V_A^2 + V_B^2 - 2V_AV_B\cos 180^o}$

$\cos 180^o = -1$

$V_{AB} = V_{BA} = \sqrt{V_A^2 + V_B^2 + 2V_AV_B}$

$= \sqrt{(V_A + V_B)^2}$

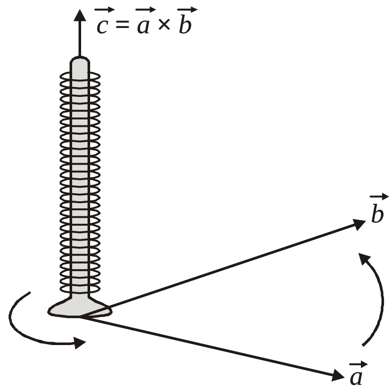
Therefore, $V_{AB} = V_{BA} = \pm(V_A + V_B)$

However, relative speeds cannot be negative.

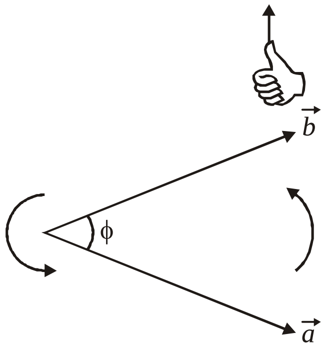
Therefore, $V_{BA} = V_{AB} = V_A + V_B$

i.e., the relative speed between two bodies moving in opposite directions is equal to the sum of the individual speeds.

**Cross Product or Vector Product of Two Vectors :** The vector product of two vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$ and is another vector $\vec{c}$ where $\vec{c} = \vec{a}\times\vec{b}$. The magnitude of is defined by $|\vec{c}| = c = ab\sin\phi$ ... (i)

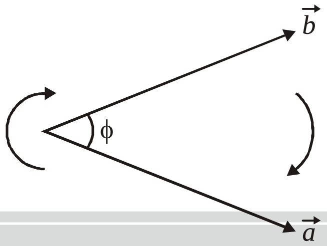



The direction of $\vec{c}$ is that in which a right handed screw advances when turned from $\vec{a}$ to $\vec{b}$




$\uparrow\ \vec{c} = \vec{a} \times \vec{b}$

The direction of $\vec{c}$ can also be obtained from the right hand rule



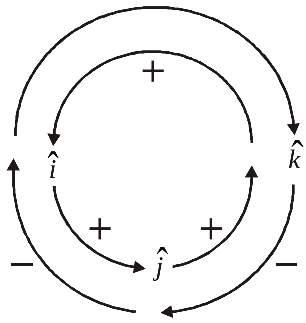

$\downarrow\ \vec{c} = \vec{b} \times \vec{a}$

The vector product changes sign when the order of the factors is reversed : $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Where, $\phi$ is the angle between *a* and *b*, the direction of $\vec{c}$, the vector product of $\vec{a}$ and $\vec{b}$ is defined to be perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. To understand the direction of the vector $\vec{c}$, let us refer to the figure. Imagine rotating a right handed screw whose axis is perpendicular to the plane formed by *a* and *b* so as to twist it from a to b through the angle $\pi$ between them. Then the direction of advance of the screw gives the direction of the vector product $\vec{a} \times \vec{b}$. Another way of determining the direction of the vector product is the right hand rule. If the right hand is held so that the curled fingers follow the rotation of $\vec{a}$ into $\vec{b}$, the extended right thumb will point in the direction of $\vec{c}$.

Note that $\vec{b} \times \vec{a}$ is not the same vector as $\vec{a} \times \vec{b}$, so that the orders of factors in a vector product is important. This is not true for scalars because, the order of factors in algebra or arithmetic does not affect the resulting product. Actually, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.






Cross product is anti-commutative, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$.

We know that $\vec{A} \times \vec{B} = \vec{C} = \vec{A}\,\vec{B}\,\hat{n}\sin\phi$, where $\hat{n}$ is a unit vector which points along the direction of vector $\vec{C}$.

Therefore, $\vec{B} \times A = -BA\,\hat{n}\sin(-\phi) = -BA\,\hat{n}\sin\phi$

Therefore, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Extending this to unit vectors, $\hat{i} \times \hat{j} = -\hat{j} \times \hat{i}$ ; $\hat{j} \times \hat{k} = -\hat{k} \times \hat{j}$ ; $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k}$.

## SOLVED EXAMPLES

**1.** Find a unit vector along vector $\vec{A} = 3\hat{i} + 2\hat{j} + 5\hat{k}$

**Soln.** We know unit vector along any vector $\vec{A}$ is given as $\hat{n} = \dfrac{\vec{A}}{A}$

Here, $A = \left|\vec{A}\right| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{9+4+25} = \sqrt{38}$

Therefore, $\hat{n} = \dfrac{3\hat{i} + 2\hat{j} + 5\hat{k}}{\sqrt{38}}$.

**2.** Find the values of '$a$' for which the vectors $\vec{A} = a\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{B} = 2a\hat{i} + a\hat{j} - 4\hat{k}$ are perpendicular to each.

**Soln.** $\vec{A} = a\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{B} = 2a\hat{i} + a\hat{j} - 4\hat{k}$

We know, if two vectors are perpendicular to each, then

$$\vec{A}\cdot\vec{B} = 0 \Rightarrow \left(a\hat{i} - 2\hat{j} + \hat{k}\right)\cdot\left(2a\hat{i} + a\hat{j} - 4\hat{k}\right) = 0$$

$$\Rightarrow \quad 2a^2 - 2a - 4 = 0$$

$$a^2 - a - 2 = 0$$

$$\Rightarrow \quad (a-2)\,(a+1) = 0$$

$$\Rightarrow \quad a = 2, -1.$$

**3.** Find the angle between the two vectors $\vec{A} = 2\hat{i} - 3\hat{j} + a\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} + 2\hat{k}$.

**Soln.** We know, if $\theta$ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$.

Here, $\quad A = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7,$

and $\quad \vec{B} = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3,$

and $\quad \vec{A}\cdot\vec{B} = \left(2\hat{i} - 3\hat{j} + 6\hat{k}\right)\cdot\left(\hat{i} + 2\hat{j} + 2\hat{k}\right) = 2 - 6 + 12 = 8.$

$$\therefore \quad \cos\theta = \frac{8}{7\times 3} = \frac{8}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{21}\right).$$

**4.** Find a vector which is perpendicular to both $\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = -6\hat{i} + 2\hat{j} - 3\hat{k}$.

**Soln.** We know, vector which is perpendicular to both $\vec{A}$ and $\vec{B}$ is given by $\vec{A}\times\vec{B}$.

$$\therefore \quad \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ -6 & 2 & -3 \end{vmatrix} = \hat{i}\,(-3-2) - \hat{j}\,(-6-6) + \hat{k}\,(4+6) = -5\hat{i} + 12\hat{j} + 10\hat{k}.$$

**5.** Find a unit vector perpendicular to the plane of vectors $\vec{A} = 2\hat{i} + 4\hat{j}$ and $\vec{B} = -4\hat{i} + 4\hat{k}$.

**Soln.** We know, vector which is perpendicular to the plane of of $\vec{A}$ and $\vec{B}$ is $\vec{A}\times\vec{B}$. Hence, unit vector in the direction of $\vec{A}\times\vec{B}$ is

$$\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

Here, $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 0 \\ -4 & 0 & 4 \end{vmatrix}$$

$$\hat{i}(16-0) - \hat{j}(8-0) + \hat{k}(0+8) = 16\hat{i} - 8\hat{j} + 8\hat{k}$$

$\therefore$ $$|\vec{A}\times\vec{B}| = \sqrt{16^2+8^2+8^2} = \sqrt{284}$$

$\therefore$ $$\hat{n} = \frac{16\hat{i} - 8\hat{j} + 8\hat{k}}{\sqrt{284}}.$$

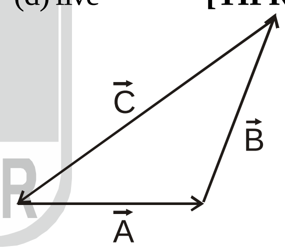
**6.** All of the following are vector quantities EXCEPT **[JNU Life Sc. 2004]**
(a) force (b) velocity (c) acceleration (d) power

**Soln.** Force, velocity and acceleration are all vectors. Power is a scalar quantity.

**Correct option is (d)**

**7.** The minimum number of unequal non-zero length vectors which can add up to give a zero resultant is
(a) two (b) three (c) four (d) five **[TIFR 2016]**

**Soln.** Let us consider three unequal non-zero vectors such that




$\vec{A}+\vec{B} = -\vec{C}$ (By triangle law of addition)

$\Rightarrow \vec{A}+\vec{B}+\vec{C} = 0$

**Correct option is (b)**

**8.** In a right-handed Cartesian frame of reference a 180º rotation about the X-axis followed by a 180º rotation about the Y-axis is equivalent to **[TIFR 2016]**
(a) an inversion operation through the origin
(b) mirror reflection down the X-Y plane
(c) 180º rotation about the Z axis
(d) 90º rotation about the Z axis

**Soln.** Let P be a point with co-ordinates $(x, y, z)$

Rotation of 180° about *X*-axis: $P(x,y,z) \longrightarrow P'(x,-y,-z)$

Rotation of 180° about *Y*-axis: $P'(x,-y,-z) \longrightarrow P''(-x,-y,z)$

The transformation $P(x,y,z) \longrightarrow P''(-x,-y,z)$ can be obtained by a 180° rotation about the *Z*–axis.

**Correct option is (c)**

**9.** Two forces of 7 Newtons each acting at 45 degrees to each other will have a resultant of approximately
(a) 6 Newtons (b) 8 Newtons **[TIFR 2018]**
(c) 10 Newtons (d) 13 Newtons

**Soln.** Given: $F_1 = 7N$, $F_2 = 7N$, angle between $F_1$ and $F_2$, $\theta = 45^o$.

Resultant $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$ N $= \sqrt{7^2 + 7^2 + 2\times 7^2 \times \cos 45^o}$ N

$$= 7\sqrt{2 + 2\times\frac{1}{\sqrt{2}}}\ \text{N} = 7\sqrt{2+\sqrt{2}}\ \text{N} \simeq 13\,\text{N}$$

**Correct option is (d)**

**10.** The vectors A and B are such that $|A + B| = |A - B|$, then the angle between the two vectors will be

**(JNU Biotech, 2003)**

(a) 0º (b) 60º (c) 90º (d) 180º

**Soln.** Given: $|\vec{A}+\vec{B}| = |\vec{A} - \vec{B}|$, let $A = |\vec{A}|$, $B = |\vec{B}|$.

$\Rightarrow \sqrt{A^2 + B^2 + 2\vec{A}.\vec{B}} = \sqrt{A^2 + B^2 - 2\vec{A}.\vec{B}}$

$\Rightarrow 4AB\cos\theta = 0 \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = 90°$

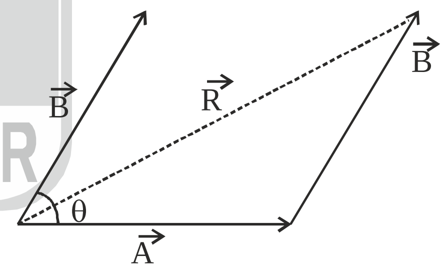
**Correct option is (c)**

**11.** Two force vectors of equal magnitude act in such a way that their resultant vector has a magnitude equal to the magnitude of either of the original forces. The angle (in degrees) between the original forces is

(a) 90 (b) 30 **(JNU Biotech, 2016)**

(c) 45 (d) 120

**Soln.** Using $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$




Given : $R = A = B = F$ (say)

or, $F^2 = F^2 + F^2 + 2F^2\cos\theta$

or, $\cos\theta = -\frac{1}{2} = \cos(90 + 30°) = \cos 120°$

or, $\theta = 120°$

**Correct option is (d)**