Chapter 11

JACOBIAN

Definition: If $u_1, u_2, u_3, ..., u_n$ be the functions of n variables $x_1, x_2, x_3, ..., x_n$ then the Jacobian of $u_1, u_2, ...u_n$ with respect to $x_1, x_2, x_3, ...x_n$ is

$$\frac{\partial(u_1,u_2,u_3,.........,u_n)}{\partial(x_1,x_2,x_3,.......,x_n)} = \mathrm{J}(u_1,u_2,u_3,.....u_n) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_4}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Example 1. If $x = r\cos\theta,\ y = r\sin\theta,$ then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Soln: $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Again we have, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2x = \cos\theta\ ; \qquad \frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2y = \sin\theta\ ;$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}.\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}\ ; \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\therefore \qquad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix} = 1/r\left(\cos^2\theta + \sin^2\theta\right) = 1/r$$

Example 2. If $u_1 = \frac{x_2 x_3}{x_1}$, $u_2 = \frac{x_3 x_1}{x_2}$ and $u_3 = \frac{x_1 x_2}{x_3}$, then prove that, $J(u_1, u_2, u_3) = 4$

Soln: We have $J(u_1,u_2,u_3) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & x_1x_2 \end{vmatrix} = \frac{(x_2x_3)(x_3x_1)(x_1x_2)}{x_1^2x_2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

Example 3. *If* $y_1 = (1 - x_1)$, $y_2 = x_1(1 - x_2)$, $y_3 = x_1 x_2 (1 - x_3)$,....

$y_n = x_1 x_2 ... x_{n-1}(1 - x_n)$, then prove that $\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ...x_n)} = (-1)^n x_1^{n-1} x_2^{n-2} .....x_{n-1}$

Soln: $\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial y_1}{\partial x_1} . \frac{\partial y_2}{\partial x_2} ..... \frac{\partial y_n}{\partial x_n} = (-1).(-x_1).(-x_1x_2).....(-x_1x_2.....x_{n-1})$

$$= (-1)^n x_1^{n-1}.x_2^{n-2}......x_{n-1}$$

Jacobian of Implicit functions:

If $u_1, u_2, u_3, ...u_n$ and $x_1, x_2, ...x_n$ are implicitly connected by n equations as:

$$f_1(u_1, u_2, ...u_n, x_1, x_2, ..., x_n) = 0$$

$$f_2(u_1, u_2, ..., u_n, x_1, x_2, ...x_n) = 0$$

....

....

$$f_n(u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0$$

then $$\frac{\partial(f_1, f_2, ...f_n)}{\partial(u_1, u_2, ...u_n)} . \frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\partial(f_1, f_2, ..., f_n)}{\partial(x_1, x_2, ...x_n)}$$

Note : The above result is a generalised result of the elementary theorem $\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$

Example 4. If $x + y + z = u$, $y + z = uv$, $z = u\,v\,w$, find the value of the Jacobian of x, y, z with respect to u, v, w.

Soln: Let, $f_1 = x + y + z - u = 0$

$$f_2 = y + z - uv = 0$$

$$f_3 = z - uvw = 0$$

Then, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \div \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

$$= -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -(-1)(-u)(-uv) \div (1)(1)(1) = u^2 v$$

Jacobian of function of function:

Theorem : If $u_1, u_2, u_3, ...u_n$ are functions of $y_1, y_2, y_3, ..., y_n$ and $y_1, y_2, y_3, ..., y_n$ are functions of $x_1, x_2, x_3, ...x_n$ then,

$$\frac{\partial(u_1,u_2,....,u_n)}{\partial(x_1,x_2,.....,x_n)} = \frac{\partial(u_1,u_2,....,u_n)}{\partial(y_1,y_2,....,y_n)} . \frac{\partial(y_1,y_2,....,y_n)}{\partial(x_1,x_2,.....,x_n)}$$

Example 5. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$

Soln: Let, $f_1 \equiv u^3 + v^3 + w^3 - x - y - z = 0$, $f_2 \equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0$

and $f_3 \equiv u + v + w - x^2 - y^2 - z^2 = 0$

We have, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 . \dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$= -\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = -\begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2b & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \div 6\begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix} = \frac{(y-z)(z-x)(x-y)}{(u-v)(u-w)(w-u)}$$

Theorem: Let $u_1, u_2, ..., u_n$ be functions of n independent variables $x_1, x_2, ...x_n$. The necessary and sufficient condition that the function be connected by a relation $f(u_1, u_2, ..., u_n) = 0$ is that the Jacobian $\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}$ vanishes identically.

Example 6. Show that the functions: $u = 3x + 2y - z, v = x - 2y + z$ and $w = x(x + 2y - z)$ are not independent, and find the relation between them.

Soln: If the given functions u, v, w are not independent, then we must have $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ equal to zero, or

$$J = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -1 \\ (2x + 2y - z) & 2x & -x \end{vmatrix} = 0$$

Adding R_2 to R_1, we get

Since, $J = 0$, hence the functions are not independent. Now we have to find the relation between u, v, w.

Clearly, $u + v = 4x$ and $u - v = 2(x + 2y - z)$

$\therefore$ $$(u + v).(u - v) = 8x(x + 2y - z) = 8w \Rightarrow u^2 - v^2 = 8w$$

Example 7. If $u = \frac{x + y}{z}, v = \frac{y + z}{x}, w = \frac{y(x + y + z)}{xz}$, show that u, v, w are not independent and find the relation between them.

Soln: We have

$$J = \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & -\frac{(x + y)}{z^2} \\ -\frac{(y + z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ -\frac{y^2 - yz}{x^2 z} & \frac{x + 2y + z}{xz} & \frac{-xy - y^2}{xz^2} \end{vmatrix}$$

taking, $\frac{1}{z^2}, \frac{1}{x^2}$ and $\frac{1}{x^2 z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J = \frac{1}{x^4 z^4} \begin{vmatrix} z & z & -(x + y) \\ -(y + z) & x & x \\ -(y^2 z + yz^2) & \frac{x + 2y + z}{xz} & \frac{-xy - y^2}{xz^2} \end{vmatrix}$$

taking $\frac{1}{z^2}, \frac{1}{x^2}$ and $\frac{1}{x^2 z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J = \frac{1}{x^4 z^4} \begin{vmatrix} z & z & -(x + y) \\ -(y + z) & x & x \quad z \\ -(y^2 z + yz^2) & x^2 z + 2xyz + xz^2 & -(x^2 y + xy^2) \end{vmatrix}$$

Applying $C_2 - C_1$ and $C_3 - C_1$,

$$= \frac{1}{x^4 z^4} \begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x^2+y^2+2xy)+z^2(x+y) & -xy(x+y)+yz(y+z) \end{vmatrix}$$

$$= \frac{1}{x^4 z^4} \begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & (x+y+z) & (x+y+z) \\ -yz(y+z) & z(x+y)(z+y+z) & (x+y+z)(yz-xy) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4 y^4} \begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & (yz-xy) \end{vmatrix}$$

Applying $C_1 + zC_3$, we get

$$= \frac{(x+y+z)^2}{x^4 z^4}(-1) \begin{vmatrix} -y & 1 \\ -yz(y+x) & z(x+y) \end{vmatrix} = \frac{(x+y+z)^2}{x^4 z^4}\left[-yz(x+y)+yz(x+y)\right] = 0$$

Since $J = 0$, hence the given functions are not independent.

Again. $uv = \dfrac{xy + y^2 + yz + zx}{zx} = \dfrac{y(x+y+z)}{xz} + 1 = w + 1$

$\therefore$ $uv = w + 1$ is the required relation between them.

Example 8. Consider the integral $\iiint\limits_V f(x, y, z)\, dV_1 = \iiint\limits_V f(r, \theta, \phi)\, dV$, if $dV_1 = dx\, dy\, dz$ and $dV_2 = J\, dr\, d\theta\, d\phi$, then the value of J is

(a) $r \sin\theta$ (b) $r^2 \sin\theta$ (c) $r \sin\phi$ (d) $r^2 \sin\phi$

Soln: Here, J is the Jacobian from Cartesian (x, y, z) to spherical polar (r, θ, ϕ)

We have, $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial r/\partial\theta & \partial r/\partial\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

On solving, this gives $= r^2 \sin\theta$. Therefore, $J = r^2 \sin\theta$.

Correct option is (b)

Example 9. If $z_1 = \dfrac{\omega_2\,\omega_3}{\omega_1}$, $z_2 = \dfrac{\omega_1\,\omega_3}{\omega_2}$, $z_3 = \dfrac{\omega_1\,\omega_2}{\omega_3}$, then the Jacobian ($J$) from $(z_1\ z_2\ z_3)$ to $(\omega_1\ \omega_2\ \omega_3)$ is equal to

(a) 3 (b) 2 (c) 4 (d) 5

Soln: $J=\dfrac{\partial(z_1\ z_2\ z_3)}{\partial(\omega_1\ \omega_2\ \omega_3)}=\begin{vmatrix}\partial z_1/\partial\omega_1 & \partial z_1/\partial\omega_2 & \partial z_1/\partial\omega_3\\ \partial z_2/\partial\omega_1 & \partial z_2/\partial\omega_2 & \partial z_2/\partial\omega_3\\ \partial z_3/\partial\omega_1 & \partial z_3/\partial\omega_2 & \partial z_3/\partial\omega_3\end{vmatrix}$

$$=\begin{vmatrix}-\omega_2\,\omega_3/\omega_1^2 & \omega_3/\omega_1 & \omega_2/\omega_1\\ \omega_3/\omega_2 & -\omega_1\,\omega_3/\omega_2^2 & \omega_1/\omega_2\\ \omega_2/\omega_3 & \omega_1/\omega_3 & -\omega_1\,\omega_2/\omega_3^2\end{vmatrix}$$

$$=\frac{1}{\omega_1^2\ \omega_2^2\ \omega_3^2}\begin{vmatrix}-\omega_2\,\omega_3 & \omega_1\,\omega_3 & \omega_1\,\omega_2\\ \omega_2\,\omega_3 & -\omega_1\,\omega_3 & \omega_1\,\omega_2\\ \omega_2\,\omega_3 & \omega_1\,\omega_3 & -\omega_1\,\omega_2\end{vmatrix}$$

$$=\frac{(\omega_2\,\omega_3)\,(\omega_1\,\omega_3)\,(\omega_1\,\omega_2)}{\omega_1^2\ \omega_2^2\ \omega_3^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4$$

Therefore, $J=4$.

Correct option is (c)

Example 10. If $u=x^2-y^2,\ v=2xy,\ x=r\cos\theta,\ y=r\sin\theta$, then the value of Jacobian $J=\dfrac{\partial(u,v)}{\partial(r,\theta)}$ is

(a) $\dfrac{1}{2}\left(x^2+y^2\right)^{3/2}$ (b) $2\left(x^2+y^2\right)^{1/2}$ (c) $4\left(x^2+y^2\right)^{3/2}$ (d) $4\left(x^2+y^2\right)^{1/2}$

Soln: We have $J=\dfrac{\partial(u,v)}{\partial(r,\theta)}=\dfrac{\partial(u,v)}{\partial(x,y)}\times\dfrac{\partial(x,y)}{\partial(r,\theta)}$

Here, $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4\left(x^2+y^2\right)$

and $\dfrac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$

Therefore, $J=4\left(x^2+y^2\right)r=4\left(x^2+y^2\right)^{3/2}$.

Correct option is (c)

Example 11. If $u=xyz,\ v=x^2+y^2+z^2,\ w=x+y+z$, then the value of Jacobian, $J=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is

(a) $-2\,(x-y)\,(y-z)\,(z-x)$

(b) $-\dfrac{1}{2\,(x-y)\,(y-z)\,(z-x)}$

(c) $-\dfrac{(x-y)\,(y-z)\,(z-x)}{2}$

(d) None of these

Soln: $J=\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y & \partial u/\partial z\\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z\\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z\end{vmatrix}=\begin{vmatrix}yz & xz & xy\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}$

$$=\begin{vmatrix}z(y-x) & x(z-y) & xy\\ -2(y-x) & -2(z-y) & 2z\\ 0 & 0 & 1\end{vmatrix}\quad [C_1\to C_1-C_2 \text{ and } C_2\to C_2-C_3]$$

$$=(y-x)(z-y)\begin{vmatrix}z & x & xy\\ -2 & -2 & 2z\\ 0 & 0 & 1\end{vmatrix}$$

$$=(x-y)(y-z)\{-2z+2x\}$$

$$=-2(x-y)(y-z)(z-x)$$

Therefore, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=-\dfrac{1}{2(x-y)(y-z)(z-x)}$.

Correct option is (b)

Example 12. Let x and y are Cartesian variable which transformed to another variable u and v such that $x=2u+3v$ and $y=2u-3v$, then the value of $|J|$ is __________. ($|J|$ denotes the modulus of J).

Soln: $J=\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}2 & 3\\ 2 & -3\end{vmatrix}=-6-6=-12$

Hence, $|J|=12$.

Correct answer is (12)