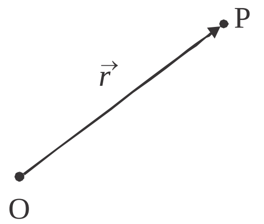
Chapter 1

Velocity and Accceleration in Different Coordinate system

In physics basic laws are first introduced for a point partile and then laws are extended to system of particles or continuous bodies. Therefore, we also begin the discussion with point particle and later on we will study collection of particles or rigid body.

To write equations governing the dynamics of a aprticle we need its position vectors, velocity, acceleration etc. Therefore we first introduce these elementaty concepts.



Position vector:
It is a vector directed from some point to location of particle. In the figure shown $\vec{r}$ is position vector of particle 'P' with respect to point 'O'. If we specify the coordinate of particle then position vector can be expressed in terms of coordinates and unit vectors used in that coordinate system.

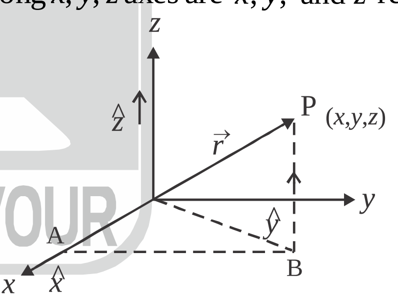
In cartesian coordinate system:
Coordinates of particle are written as (x, y, z) and unit vectors along x, y, z axes are $\hat{x}, \hat{y},$ and $\hat{z}$ respectively. Therefoer, from figure,

$$OA = x,\ AB = y,\ BP = z$$

and $\quad \vec{OP} = \vec{OB} + \vec{BP}$

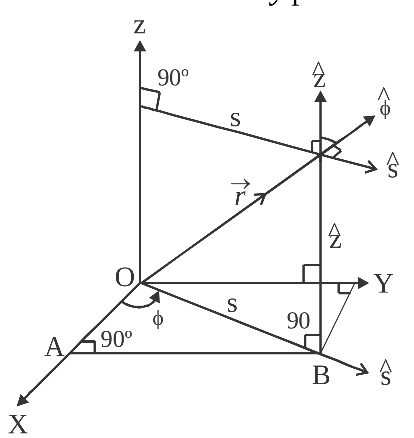
$$\vec{r} = \vec{OA} + \vec{AB} + \vec{BP} = x\hat{x} + y\hat{y} + z\hat{z}$$

$\therefore \quad \boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$



Unit vectors are taken in the directions in which coordinates increase.

In cylindrical system coordinates of particle are written as (s, ϕ, z) and unit vectors along the increasing direction of coordinates are $(\hat{s}, \hat{\phi}, \hat{z})$. s is perpendicular distance of particle from z-axis, ϕ is its angular position with respect to x-axis and z is its distance above x-y plane.



Therefore, from figure $OB = s$, $\angle AOB =$, $BP = z$ and $\vec{OP} = \vec{OB} + \vec{BP}$

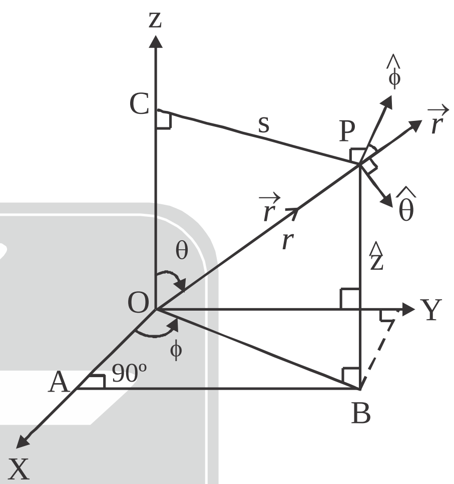
$$\boxed{\vec{r} = s\hat{s} + z\hat{z}}$$

relation with cartesian coordinates

$$\boxed{x = s\cos\phi,\ y = s\sin\phi,\ z = z} \text{ and } \boxed{\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y},\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y},\ z = \hat{z}}$$

In spherical polar coordinates system, coordinates of particle are written as (r, θ, ϕ) and unit vector in increasing direction of coordinates are $(\hat{r}, \hat{\theta} \text{ and } \hat{\phi})$. r is the distance of particle from origin, θ and ϕ are angular position with respect to z and x axes respectively.

From figure $OP = r$, $\angle COP = \theta$, $\angle AOB = \phi$ and $\vec{OP} = r\hat{r}$



$\therefore \quad \boxed{\vec{r} = r\hat{r}}$

$OB = CP = r\sin\theta, \qquad BP = r\cos\theta$

$\therefore \quad AB = OB\sin\phi = r\sin\theta.\sin\phi$ and

$OA = OB\cos\phi = r\sin\theta\cos\phi$

Relation with cartesian coordinate,

$$\boxed{x = r\sin\theta.\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta}$$

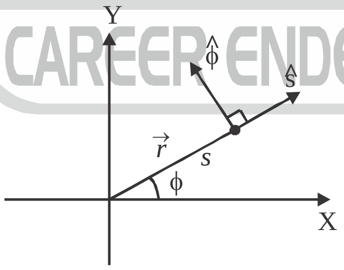
And $\quad \hat{r} = \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}$

$$\hat{\theta} = \cos\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) - \sin\theta\hat{z}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

If motion of a particle is confined in one plane then only two coordinates are required to describe its position. We can either use cartesian coordinates (x, y) or plane polar coordinates (s, ϕ).

Thus if a particle is moving on a plane then its position vector can be written as



$$\vec{r} = x\hat{x} + y\hat{y}$$

Or, $\quad \vec{r} = s\hat{s}$ in (plane polar coordinate)

Plane polar coordinates (s, ϕ) are the same coordinates which are used in cylindrical coordinates system.

Notice that, $\hat{x}, \hat{y}$ and $\hat{z}$ have a fixed direction as they are along the x, y and z axes, whereas $\hat{r}, \hat{s}, \hat{\theta}, \hat{\phi}$ etc do not have fixed directions. Therefore, $\hat{x}, \hat{y}, \hat{z}$ are constant unit vectors but $\hat{r}, \hat{s}, \hat{\theta}, \hat{\phi}$ are not constant unit vectors.

Thus, $\quad \dfrac{d\hat{x}}{dt} = 0, \dfrac{d\hat{y}}{dt} = 0, \dfrac{d\hat{z}}{dt} = 0$ and $\dfrac{d\hat{r}}{dt} \neq 0, \dfrac{d\hat{s}}{dt} \neq 0, \dfrac{d\hat{\theta}}{dt} \neq 0, \dfrac{d\phi}{dt} \neq 0$

Derivative of unit vectors $(\hat{r}, \hat{\theta}, \hat{s}, \hat{\phi})$ can easily be found by using their relation with $(\hat{x}, \hat{y}, \hat{z})$.

For example: In plane polar or cylindrical coordinates, $\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$ and $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$

$$\therefore \quad \frac{d\hat{s}}{dt} = -\sin\phi\frac{d\phi}{dt}\hat{x} + \cos\phi\frac{d\phi}{dt}\hat{y}$$

$$= \left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)\frac{d\phi}{dt} = \hat{\phi}\dot{\phi}$$

and $$\frac{d\hat{\phi}}{dt} = -\cos\phi\frac{d\phi}{dt}\hat{x} - \sin\phi\frac{d\phi}{dt}\hat{y}$$

$$= -\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right)\frac{d\phi}{dt} = -\hat{s}\dot{\phi}$$

$$\therefore \quad \boxed{\frac{d\hat{s}}{dt} = \hat{\phi}\dot{\phi}, \qquad \frac{d\hat{\phi}}{dt} = -\hat{s}\dot{\phi}}$$

Velocity: Average velocity is defined as,

$$\vec{u}_{av} = \frac{\text{total displacement}}{\text{total time taken}} = \frac{\text{change in position vector}}{\text{total time taken}}$$

$$= \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

Instantaneous velocity (velocity at any instant of time) is defined as time derivative of position vector.

Instantaneous velocity, $\vec{v} = \frac{d\vec{r}}{dt}$.

By expressing $\vec{r}$ in different coordinate system.

In cartesian coordinate system:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Therefore, $$\vec{v} = \frac{d}{dt}\left(x\hat{x} + y\hat{y} + z\hat{z}\right) = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$\boxed{\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}}$$

Or, $v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$v_x = \dot{x} = \frac{dx}{dt}$ is component of velocity in x direction.

$v_y = \dot{y} = \frac{dy}{dt}$ is component of velocity in y direction

$v_z = \dot{z} = \frac{dz}{dt}$ is component of velocity in z direction

In plane coordinate system:

$$\vec{r} = s\hat{s}$$

Therefore, $$\vec{v} = \frac{d}{dt}\left(s\hat{s}\right) = \frac{ds}{dt}\hat{s} + s\frac{d\hat{s}}{dt}$$

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}} \qquad \left\{\because \frac{d\hat{s}}{dt} = \dot{\phi}\hat{\phi}\right\}$$

Or, $\vec{v} = v_s\hat{s} + v_\phi\hat{\theta}$

$v_s = \dot{s} = \dfrac{ds}{dt}$ is component velocity in $\hat{s}$ direction and it is called radial velocity

$v_\phi = s\dot{\phi} = s\dfrac{d\phi}{dt}$ is component velocity in $\hat{\phi}$ direction and it is called transverse velocity.

In cylindrical coordinate system:

$$\vec{r} = s\hat{s} + z\hat{z}$$

Therefore, $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(s\hat{s} + z\hat{z})$$

$$= \frac{ds}{dt}\hat{s} + s\frac{d\hat{s}}{dt} + \frac{dz}{dt}\hat{z}$$

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}} \qquad \left\{\because \frac{d\hat{s}}{dt} = \dot{\phi}\hat{\phi}\right\}$$

Acceleration : Average acceleration is defined as

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Instantaneous acceleration is defined as time derivative of velocity vector.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

By expressing $\vec{v}$ indifferent coordinatge system. We can writge acceleration of a particle in different coordinate system.

In cartesian coordinate:

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

Therefore, $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z})$$

$$\boxed{\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}}$$

or, $$\vec{a} = \left(a_x\hat{x} + a_y\hat{y} + a_z\hat{z}\right)$$

$a_x = \ddot{x} = \dfrac{d}{dt}(\dot{x}) = \dfrac{dv_x}{dt}$ is component of acceleration along x-direction.

$a_y = \ddot{y} = \dfrac{d}{dt}(\dot{y}) = \dfrac{dv_y}{dt}$ is component of acceleration along y-direction.

$a_z = \ddot{z} = \dfrac{d}{dt}(\dot{z}) = \dfrac{dv_z}{dt}$ is component of acceleration along z-direction.

Since, $a_x = \dfrac{dv_x}{dt}$, therefore, if velocity along x-direction is constant then acceleration along x-direction must be zero.

In Plane polar coordinate :

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}$$

Therefore, $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{s}\hat{s} + s\dot{\phi}\hat{\phi}\right)$$

$$= \ddot{s}\hat{s} + \dot{s}\frac{d\hat{s}}{dt} + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\frac{d\hat{\phi}}{dt}$$

$$= \ddot{s}\hat{s} + \dot{s}\left(\dot{\phi}\hat{\phi}\right) + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\left(-\dot{\phi}\hat{s}\right)$$

$$\boxed{\vec{a} = \left(\ddot{s} - s\dot{\phi}^2\right)\hat{s} + \left(2\dot{s}\dot{\phi} + s\ddot{\phi}\right)\hat{\phi}}$$

Or, $$\vec{a} = a_s\hat{s} + a_\phi\hat{\phi}$$

$a_s = \ddot{s} - s\dot{\phi}^2$ is component of acceleration along $\hat{s}$ and its called radial acceleration. Clearly, it is not equal to time derivative of radial component of velocity $\left(v_s = \dot{s}\right)$.

Therefore, if $v_s = \text{constant}$, then a_s may not be zero.

$a_\phi = 2\dot{s}\dot{\phi} + s\ddot{\phi}$ is component of acceleration in $\hat{\phi}$ direction. It is called transverse acceleration. Clearly it is also not equal to time derivative of radial component of velocity $\left(v_\phi = s\dot{\phi}\right)$

i.e. $$a_\phi \neq \frac{d}{dt}\left(v_\phi\right)$$

Therefore, if $v_\phi = \text{constant}$ then a_ϕ may not be zero.

In cylindrical coordinate system:

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

First two terms are same as in plane polar coordinate.

Therefore, $$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{s} - s\dot{\phi}^2\right)\hat{s} + \left(2\dot{s}\dot{\phi} + s\ddot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}}$$

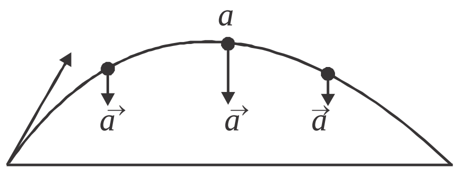
How to choose a coordinate system:

Generally speaking, dynamics of a particle can be studied using any coordinate system. However judicious selection of coordinate system makes task easy.

One dimensional cases is simple because we just need one coordinate which we can take either x or y or z.

However in two dimensional cases one has to be contious. In most of the two dimensional problems if acceleration of the particle is directed towards a point then plane polar coordinate is conveninet to use. However, if accleration is not directed towards a point then cartesian coordinate is right choice.

For example : A projectile motion in which particle is projected with a small speed at some angle with horizontal. It moves with constant acceleration which is directed downward. In this case acceleration is not directed towards a point. Therefore, we can conveniently use cartesian coordinates.



In planetary motion, acceleration of planet is always directed towards centre of the sun. Therefore in this case plane polar coordinate makes calculation simple.

If a particle is attached to a point with a string or spring then also generally polar coordinate is helpfull. For example, in case of simple pendulum.

Formulae of kinematics (uniformly accelerated motions): When a constant force acts on a particle its acceleration is constant. If u_x, v_x be initial and final velocity of a particle along x-direction and a_x be its acceleration along x direction then.

$$\left.\begin{aligned} v_x &= u_x + a_x t \\ v_x^2 &= u_x^2 + 2a_x x \\ x &= u_x t + \tfrac{1}{2} a_x t^2 \end{aligned}\right\} \text{if } a_x = \text{constant}$$

where x is displacement along x-direction and t is time taken. We can write similar relations for y and z direction also.

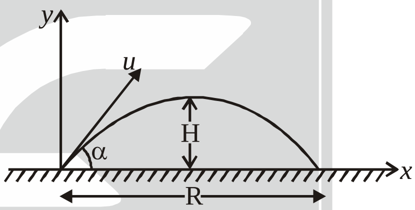
Note: If acceleration of particle is not constant then we cannot use formulae of kinematics. In that case we start with either definition of velocity or definition of acceleration i.e.

$$v_x = \frac{dx}{dt} \quad \text{or} \quad a_x = \frac{dv_x}{dt}$$

we may also have to use $a_x = \dfrac{F_x}{m}$ and $\dfrac{dv_x}{dt} = \dfrac{dv_x}{dx} \cdot \dfrac{dx}{dt} = \dfrac{v_x dv_x}{dx}$ as the case may be

Projectile motion: If u be the projection speed and α be angle of projection then equation of path of projectile is,

$$y = x \tan\alpha - \frac{gx^2}{2u^2 \cos^2\alpha}$$



Range, $R = \dfrac{u^2 \sin 2\alpha}{g}$

Maximum height, $H = \dfrac{u^2 \sin^2\alpha}{2g}$

Total time of flight, $T = \dfrac{2u \sin\alpha}{g}$

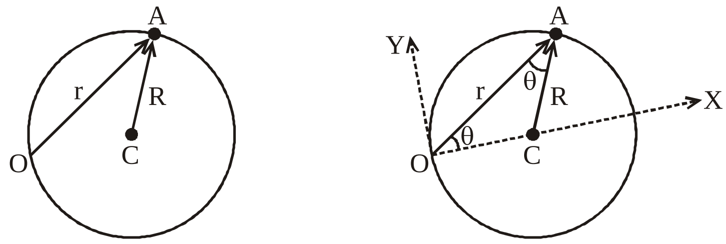
Coordinate of projectile at any time $t < T$.

$$x = u\cos\alpha\, t, \; y = u \sin\alpha\, t - \frac{1}{2} g t^2$$

Note: Above results for projectile are applicable only if projectile is thrown from ground and it finally lands on ground. And projection speed is small so that projectile does not go too high. If projection speed is not small then height will be large and in that case we need to consider variation of acceleration due to gravity.

SOLVED PROBLEMS

1. A particle A moves along a circle of radius $R = 50$ cm so that its radius vector r relative to point O (figure (a)) rotates with the constant angular velocity $\omega = 0.40$ rad/sec. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.



(a) **(b)**

Soln. Consider X and Y axes as shown in (figure (b)). Using sine law in triangle CAO, we get $\dfrac{r}{\sin(\pi-2\theta)} = \dfrac{R}{\sin\theta}$

or $$\frac{r}{2\sin\theta\cos\theta} = \frac{R}{\sin\theta}$$

$\therefore \quad r = 2R\cos\theta$

Now $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} = 2R\cos^2\theta\,\mathbf{i} + 2R\cos\theta\sin\theta\,\mathbf{j}$

Now $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -4R\cos\theta\sin\theta\dfrac{d\theta}{dt}\mathbf{i} + 2R\cos2\theta\dfrac{d\theta}{dt}\mathbf{j} = -2R\sin2\theta\,\omega\,\mathbf{i} + 2\theta\omega\,\mathbf{j}$

$\therefore \quad |\mathbf{v}| = 2\omega R$

Further

$$\mathbf{a} = \frac{dv}{dt} = 4R\cos2\theta\,\omega\frac{d\theta}{dt}\mathbf{i} - 4R\omega\sin2\theta\frac{d\theta}{dt}\mathbf{j} = -4R\omega^2\cos2\theta\,\mathbf{i} - 4R\omega^2\sin2\theta\,\mathbf{j}$$

$$|\mathbf{a}| = 4R\omega^2.$$

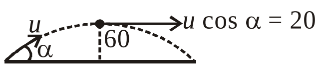
2. After being hit, a golf ball reaches a maximum height of 60m with speed of 20m/s. Right after being hit the speed of the ball is (take g = 10 m/s^2) **[GP-2008]**

(a) 100m/s (b) 80m/s (c) 60m/s (d) 40m/s

Soln. (d) If u be the speed just after being hit at angle α then

$$u\cos\alpha = 20 \qquad \text{... (i)}$$

$$\frac{u^2\sin^2\alpha}{2g} = 60$$



$$\therefore \quad u^2\sin^2\alpha = 1200 \qquad \text{... (ii)}$$

on squaring (i) and adding to (ii) we get

$$u^2 = 1200 + 400$$

$$\therefore \quad u = 40$$

3. A particle starts moving with acceleration $a = \alpha - \beta v$ where α and β are constants and v is its instantaneous speed. Find velocity of particle as a function of time and also find its terminal velocity.

Soln: Acceleration is variable therefore we start with definition of acceleration

$$a = \frac{dv}{dt} \quad \text{or} \quad \alpha - \beta v = \frac{dv}{dt} \quad \text{or} \quad \int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt$$

$$-\frac{1}{\beta}\ln\left(\frac{\alpha - \beta v}{\alpha}\right) = t \quad \text{or,} \quad 1 - \frac{\beta v}{\alpha} = e^{-\beta t} \qquad \therefore v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

as $t \to \infty$, $v = \dfrac{\alpha}{\beta}$ = constant. This is terminal velocity.

4. A projectile is thrown at an angle α with horizontal with initial speed u after what time the projectile's velocity will be perpendicular to its initial direction.

Soln. Suppose velocity after time t becomes perpendicular to initial velocity. Velocity after time t is

$$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j} \quad = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$$

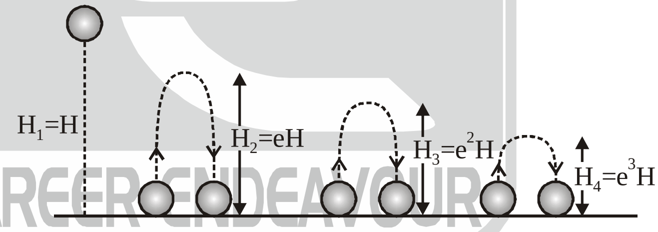
Initial velocity $\vec{u} = u(\cos\alpha\hat{i} + \sin\alpha\,\hat{j})$

$\vec{v}$ is $\perp$ to $\vec{u}$, therefore, $\vec{v}.\vec{u} = 0 \quad \therefore u\cos^2\alpha + u\sin^2\alpha - gt\sin\alpha = 0 \quad (\because\ u \neq 0)$

$$\therefore t = \frac{u}{g\sin\alpha}$$

5. A ball is dropped from a height H. It bounces back up to a height e times after hitting the ground. If e<1, after what time the ball will finally come to rest.

Soln. Ball moves under the effect of gravity. Therefore magnitude of acceleration of ball during upward or downward movement remains constant.



For first bounce

$$y = H,\ u_y = 0,\ a_y = g$$

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\therefore H = 0 + \frac{1}{2}gt^2 \qquad \therefore t = \sqrt{\frac{2H}{g}}$$

Therefore time of fall before first bounce $t_1 = \sqrt{\dfrac{2H}{g}}$

Under gravity, time for going up a certain distance is equal to time for going down. Therefore time interval between first bounce and second bounce is

$$t_2 = 2\sqrt{\frac{2H_2}{g}} = 2\sqrt{\frac{2eH}{g}}$$

Similarly $t_3 = 2\sqrt{\dfrac{2e^2H}{g}}$, $t_4 = 2\sqrt{\dfrac{2e^3H}{g}}$,.....

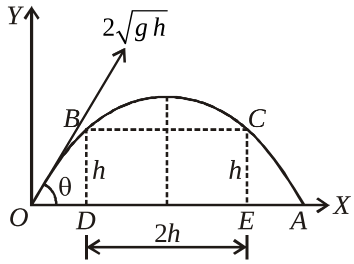
Therefore, total time elapsed before ball stops is

$$T = t_1 + t_2 + t_3 + ...$$

$$= \sqrt{\frac{2H}{g}}\left[1 + 2\left(e^{1/2} + e^{2/2} + e^{3/2} + ...\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2\frac{e^{1/2}}{1 - e^{1/2}}\right] = \sqrt{\frac{2H}{g}}\left(\frac{1+\sqrt{e}}{1-\sqrt{e}}\right)$$

6. A particle is projected with velocity $2\sqrt{(h\,g)}$ so that it just clears two walls of equal height h which are at a distance $2h$ from each other. Show that the time of passing between the walls is $2\sqrt{(h/g)}$.

Soln. The situation is shown in figure.



Suppose the particle is at a height h after time t. Then,

$$h = 2\sqrt{g\,h}\,\sin\theta\cdot t - \frac{1}{2}g\,t^2$$

or $\quad g\,t^2 - 4\sqrt{g\,h}\,\sin\theta\cdot t + 2h = 0 \qquad \text{... (i)}$

This equation gives the two values of time at which the height of the particle is h. Let these times be t_1 and t_2 respectively. Then,

$$(t_1 - t_2)^2 = (t_1 + t_2) - 4t_1\,t_2$$

From equation (i)

$$t_1 + t_2 = \frac{4\sqrt{(g\,h)}\,\sin\theta}{g} \text{ and } t_1\,t_2 = \frac{2h}{g}$$

$\therefore \quad (t_1 - t_2)^2 = \dfrac{16gh\sin^2\theta}{g^2} - \dfrac{8h}{g} = \dfrac{8h}{g}[2\sin^2\theta - 1]$

or $\quad (t_1 - t_2) = \sqrt{\left(\dfrac{8h}{g}\right)}\,[2\sin^2\theta - 1]^{1/2} \qquad \text{... (ii)}$

During this time, the horizontal distance moved is $2h$. The horizontal velocity being $2\sqrt{(g\,h)}\,\cos\theta$.

Using $s = ut$, we have

$$2h = 2\sqrt{(g\,h)}\,\cos\theta\cdot t$$

or $\quad 2h = 2\sqrt{(g\,h)}\,\cos\theta\times\sqrt{(8h/g}\,[2\sin^2\theta - 1]^{1/2}$

Squaring $1 = 8\cos^2\theta\,(1 - 2\cos^2\theta)$

or $\quad 16\cos^4\theta - 8\cos^2\theta + 1 = 0,$

or $\quad (4\cos^2\theta - 1)^2 = 0.$

$\therefore \quad \cos^2\theta = \dfrac{1}{4}$ and $\sin^2\theta = \dfrac{3}{4}. \qquad \text{... (iii)}$

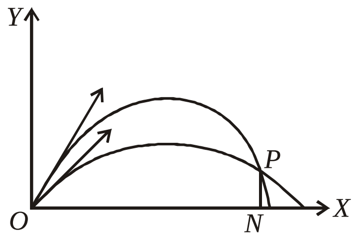
Substituting these values in equation (ii) we have

$$(t_1 - t_2) = \sqrt{[8\,h/g]}\left[2\cdot\frac{3}{4} - 1\right]^{1/2}$$

or $\quad t_1 - t_2 = 2\sqrt{(h/g)}.$

7. Two particles are projected from a point at the same instant with velocities whose horizontal and vertical components are u_1, v_1 and u_2, v_2 respectively. Prove that the interval between their passing through the other common point of their path is $\dfrac{2(v_1 u_2 - v_2 u_1)}{g(u_1+u_2)}$.

Soln. The situation is shown in figure.



Let the two particles pass through the common point P. Again let, t_1 and t_2 be the time taken by the particles to go to P from O.

Along X-axis, we have

$$u_1 t_1 = u_2 t_2 \qquad \text{... (i)}$$

In the vertical direction

$$v_1 t_1 - \frac{1}{2} g t_1^2 = PN = v_2 t_2 - \frac{1}{2} g t_2^2 \qquad \text{... (ii)}$$

From equation (ii), we have

$$t_1^2 - t_2^2 = 2/g \cdot (v_1 t_1 - v_2 t_2) \qquad \text{... (iii)}$$

But from equation (i), $\dfrac{u_1}{u_2} = \dfrac{t_2}{t_1}$.

Addition one to both sides, we have $\dfrac{u_1+u_2}{u_2} = \dfrac{t_2+t_1}{t_1}$.

$$\therefore \quad (t_1+t_2) = \frac{(u_1+u_2)}{u_2} \times t_1 \qquad \text{... (iv)}$$

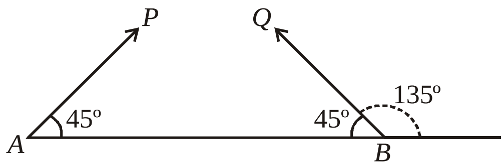
From equation (iii), we get

$$(t_1+t_2)(t_1-t_2) = \frac{2}{g} \cdot (v_1 t_1 - v_2 t_2)\,(t_1-t_2) \times t_1 = \frac{2}{8}\left(v_1 t_1 - v_2 \frac{u_1}{u_2} \cdot t_1\right)$$

$$(t_1+t_2)\frac{(u_1+u_2)}{u_2} = \frac{2}{g}\left(v_1 - v_2 \frac{u_1}{u_2}\right)(t_1+t_2) = \frac{2}{g} \cdot \frac{u_2}{(u_1+u_2)} \times \frac{(v_1 u_2 - v_2 u_1)}{u_2}$$

or $\quad (t_1 - t_2) = \dfrac{2}{g} \dfrac{(v_1 u_2 - v_2 u_1)}{(u_1+u_2)}$.

8. Particles P and Q of mass 20 gm and 40 gm respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q make 45º and 135º angles respectively with the horizontal as shown in the figure. Each particle has an initial speed of 49 m/sec. The separation AB is 245 m. Both particles travel in the same vertical plane and undergo a collision. After the collision P retraces its path. Determine the position of Q when it hits the ground. How much time after the collision does the particle Q take to reach the ground. Take g = 9.8 m/sec^2.



Soln. The horizontal velocity of either particle is $49\cos 45^\circ = 49/\sqrt{2}$ m/sec. Initially, the vertical component of velocity of either particle $= 49\sin 45^\circ = 49/\sqrt{2}$ m/sec. The horizontal distance travelled by each particle at the time of collision = 245/2 metre. The vertical component of the velocity of each particle at the time of collision is

$$\frac{49}{\sqrt{2}} - 9.8 \times \frac{245}{2} \times \frac{\sqrt{2}}{49} = 0.$$

At the time of collision, the velocity of each particle is horizontal and is equal to $49/\sqrt{2}$ m/sec.

Let v_p and v_q be the velocities after collision, then

$$0.02 \times \frac{49}{\sqrt{2}} - 0.04 \times \frac{49}{\sqrt{2}} = 0.02\, v_p + 0.04\, v_q.$$

As P retraces its path, $v_p = -\frac{49}{\sqrt{2}}$.

$$\therefore \quad \frac{49}{\sqrt{2}}(0.02 - 0.04) = -0.02 \times \frac{49}{\sqrt{2}} + 0.04\, v_q$$

The particle falls vertically downward from the point of collision, i.e., at a distance of 122.5 metre from A.

Let t be the time taken, then $\frac{49}{\sqrt{2}} = 9.8 \times t$.

Since collision does not after the vertical component of velocities.

$$\therefore \quad t = \frac{49}{\sqrt{2} \times 9.8} = \frac{5}{\sqrt{2}} = 3.53 \text{ sec}.$$

9. Two shots are projected from a gun at the top of a hill with the same velocity u at angles of projection α and β respectively. If the shots strike the horizontal ground through the foot of the hill at the same point, show that the height h of the hill above the plane is given by

$$h = \frac{2u^2(1 - \tan\alpha \tan\beta)}{g(\tan\alpha + \tan\beta)^2}.$$

Soln. Let R be the horizontal range for both. Taking the point of projection as origin, the point struck is (R, h). This point satisfies the equations of both the trajectories.

$$-h = R\tan\alpha - \frac{1}{2} g \frac{R^2}{u^2 \cos^2\alpha} \qquad \text{... (i)}$$

and $$-h = R\tan\beta - \frac{1}{2} g \frac{R^2}{u^2 \cos^2\beta} \qquad \text{... (ii)}$$

From these equations, we have

$$R\tan\alpha - \frac{1}{2} g \frac{R^2}{u^2\cos^2\alpha} = R\tan\beta - \frac{1}{2} g \frac{R^2}{u^2\cos^2\beta}$$

or $\quad R(\tan\alpha - \tan\beta) = \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$

$$2u^2/g\ (\tan\alpha - \tan\beta) = R(\sec^2\alpha - \sec^2\beta)$$

or $\quad 2u^2/g\ (\tan\alpha - \tan\beta) = R[(1+\tan^2\alpha) - (1+\tan^2\beta)]$

or $\quad 2u^2/g = R(\tan\alpha + \tan\beta)$. ... (iii)

$$R = \frac{2u^2}{g(\tan\alpha + \tan\beta)}$$

Putting the value of R in equation (i), we have

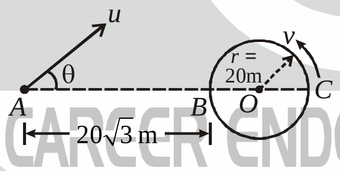
$$h = R\left[\frac{gR}{2u^2\cos^2\alpha} - \tan\alpha\right] = \frac{2u^2}{g(\tan\alpha+\tan\beta)}\left[\frac{\sec^2\alpha}{\tan\alpha+\tan\beta} - \tan\alpha\right]$$

$$= \frac{2u^2}{g(\tan\alpha+\tan\beta)^2}\left[\sec^2\alpha - \tan^2\alpha - \tan\alpha\tan\beta\right] = \frac{2u^2}{g(\tan\alpha+\tan\beta)^2}(1-\tan\alpha\tan\beta).$$

10. A particle is moving along a vertical circle of radius $r = 20$ m with a constant speed $v = 31.4$ m/s as shown in figure below. Straight line ABC is horizontal and passes through the center of the circle. A shell is fired from point A at the instant when the particle is at C. If distance AB is $20\sqrt{3}$ m and the shell collide with the particle at B, then prove

$$\tan\theta = \frac{(2n-1)^2}{\sqrt{3}},$$

where n is an integer. Further show that smallest value of θ is 30º.



Soln. As at the time of firing of the shell, the particle was at C and the shell collides with it at B, therefore the number of revolutions completed by the particle is odd multiple of half i.e., $(2n-1)/2$, where n is an integer.

Let T be the time period of the particle, then

$$T = \frac{2\pi r}{v} = \frac{2\times 3.14\times 20}{31.4} = 4 \text{ second}$$

If t be the time of flight of the shell, then t = time of $[(2n-1)/2]$ revolutions of the particle

$$= \frac{(2n-1)}{2}\times 4 = 2(2n-1) \text{ second}$$

for a projectile, the time of flight is given by $t = \frac{2u\sin\theta}{g}$.

Hence, $\frac{2u\sin\theta}{g} = 2(2n-1)$ ... (i)

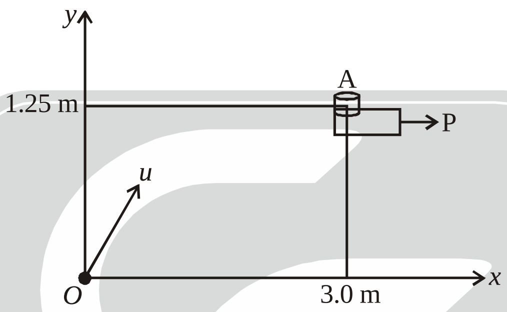
The range of the projectile is given by $R = \frac{u^2\sin 2\theta}{g}$.

Hence, $\dfrac{u^2 \sin 2\theta}{g} = 20\sqrt{3}$ ... (ii)

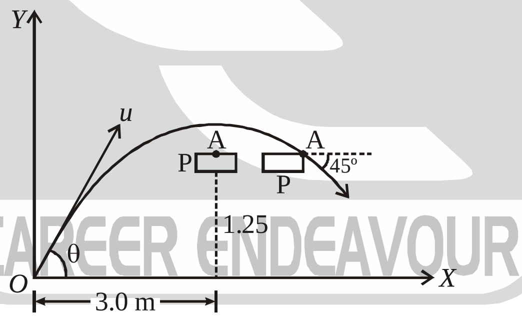
From equations (i) and (ii), we get $\tan\theta = \dfrac{(2n-1)^2}{\sqrt{3}}$.

For θ to be smallest, $n = 1$, so $\theta_{min} = 30^\circ$.

11. An object *A* is kept fixed at the point $x = 3\,\text{m}$ and $y = 1.25\,\text{m}$ on a plank *P* raised above the ground. At time $t = 0$ the plank starts moving along the $+x$ direction with an acceleration 1.5 m/s^2. At the same instant a stone is projected from the origin with a velocity *u* as shown. A stationary person on the ground observes the stone hitting the object during its downward motion at an angle of 45° to the horizontal. All the motions are in the *xy*-plane. Find *u* and the time after which the stone hits the object. Take $g = 10\ \text{m/s}^2$.



Soln. See figure.



Let θ be the angle of projectile. Further, suppose the stone hits the object at time *t*, then

$$(u\cos\theta)t = 3.0 + \frac{1}{2} \times 1.5 \times t^2 \quad \text{... (i)}$$

and $$(u\sin\theta)t - \frac{1}{2}gt^2 = 1.25 \quad \text{... (ii)}$$

As the stone hits the object *A*, moving along 45° with the horizontal during downward motion, we have

$$\tan(-45^\circ) = \left(\frac{v_y}{v_x}\right) = \frac{u\sin\theta - gt}{u\cos\theta}$$

or $$\frac{u\sin\theta - gt}{u\cos\theta} = -1$$

or $$u\sin\theta - gt = -u\cos\theta \quad \text{... (iii)}$$

From equation (i), $u\cos\theta = \frac{3.0}{t} + 0.75t$

From equation (ii), $u\sin\theta = \frac{1.25}{t} + \frac{g}{2}t$

Substituting these values in equation (iii), we get

$$\frac{1.25}{t} + 5t - 10t = -\frac{3.0}{t} - 0.75t$$

$$\frac{4.25}{t} = 4.25t \text{ or } t^2 = 1 \text{ or } t = 1\,\text{sec}.$$

From equation (i) and (ii), we get

$$u\cos\theta = 3.0 + 0.75 = 3.75 \qquad \text{... (iv)}$$

and $u\sin\theta = 6.25$ ... (v)

Dividing equation (v) equation (iv), we get

$$\tan\theta = \frac{6.25}{3.75} = \frac{5}{3} \text{ or } \theta = \tan^{-1}\left(\frac{5}{3}\right)$$

Squaring and adding equation (iv) and (v), we get

$$u^2 = (3.75)^2 + (6.25)^2 = (14.06) + (39.06)$$

$$u^2 = 53.12 \text{ or } u = 7.29 \text{ m/s}.$$

12. A rotation disc (figure below) moves in the positive direction of the *x*-axis. Find the $y(x)$ describing the positive of the instantaneous axis of rotation, if at the initial moment the axis *C* of the disc was located at the point *O* after which it moved

(a) with a constant velocity *v*, while the disc started rotating counter clockwise with a constant angular acceleration α (the initial angular velocity is equal to zero);

(b) with a constant acceleration *a* (and the zero initial velocity), the disc rotates counter clockwise with a constant angular velocity ω.

Soln. (a) When the disc rotates with constant angular velocity *v* and counter clockwise angular acceleration α :

We have the relation $\omega = \omega_0 + \alpha t = \alpha t,\ t = \frac{x}{v}$.

Now consider any point *P* of the disc at a distance *y* from the *x*-axis; i.e. $\mathbf{r} = -y\,\mathbf{j}$. An axis passing through *P* and parallel to the axis of rotation. Now the linear velocity of point *P*.

$$\mathbf{v}_p = \vec{\omega}\times\mathbf{r} \text{ and } \vec{\omega} = \alpha\frac{x}{v}\mathbf{k} = \left(\frac{\alpha x}{v}\right)\mathbf{k}\times y(-\mathbf{j}) = \frac{\alpha x}{v}y\,\mathbf{i};$$

that is, the point *P* has a velocity along the *x*-axis which is the same as the velocity of any point of the disc = *v*.

Hence, $y = \frac{v^2}{\alpha x}$: (equation of a hyperbola)

(b) When the disc rotates with constant angular velocity ω and constant acceleration *a* : In this case

$$x = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\left(\frac{2x}{a}\right)}$$

Hence, $v = at = \sqrt{2xa}$

Again $\mathbf{v}_p = \vec{\omega} \times \mathbf{r} = \vec{\omega}\mathbf{k} \times y(-\mathbf{j})$,

Therefore, $|\mathbf{v}| = \omega y$ or $\sqrt{2xa} = \omega y$ or $y = \dfrac{\sqrt{2xa}}{\omega}$, a parabola].

13. A balloon rises from rest on the ground with constant acceleration $g/8$. A stone is dropped when the balloon has risen to a height H meter. Show that the time taken by the stone to reach the ground is given by $2\sqrt{(H/g)}$.

Soln. The velocity v of the balloon when it has risen to a height H is given by $v^2 = 0 + 2 \times \dfrac{g}{8} \times H$ taking upward direction positive.

$$\therefore \qquad v = \sqrt{\left\{\frac{(gH)}{4}\right\}} = \frac{\sqrt{(gH)}}{2} \text{ m/sec.}$$

This will also be the velocity of the stone in upward direction when it is dropped.

Taking upward direction position for the stone, $-H = \dfrac{\sqrt{(gH)}}{2}t - \dfrac{1}{2}t - \dfrac{1}{2}gt^2 \quad (\because\ g = -g \text{ and } h = -H)$

or $\qquad gt^2 - \sqrt{(gH)}\,t - 2H = 0$

$$t = \frac{\sqrt{(gH)} \pm \sqrt{\{(gH) + (8gH)\}}}{2g} = \frac{\sqrt{(gH)} \pm 3\sqrt{(gH)}}{2g} = 2\sqrt{\left(\frac{H}{8}\right)} \text{ taking positive value.}$$

14. A particle is projected vertically upwards. Prove that it will be at 3/4 of its greatest height at time which are in the ratio 1 : 3.

Soln. We know that the greatest height attained $= u^2/2g$ (where u is the initial velocity). Let t be the time when the particle is at a height $\dfrac{3}{4}\left(\dfrac{u^2}{2g}\right)$.

Using the formula $s = ut + \dfrac{1}{2}gt^2$, we have $\dfrac{3}{4}\left(\dfrac{u^2}{2g}\right) = ut - \dfrac{1}{2}gt^2 \Rightarrow t^2 - \dfrac{2u}{g}t + \dfrac{6u^2}{8g^2} = 0.$

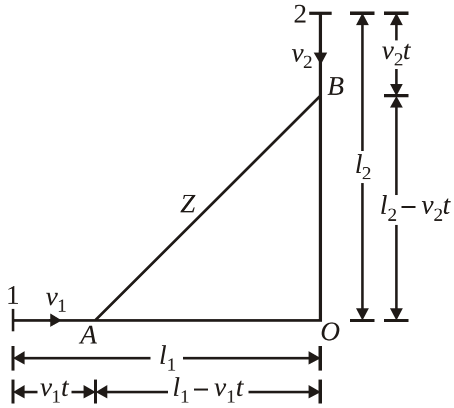
Solving for t, we have $t = \dfrac{\dfrac{2u}{g} \pm \sqrt{\left(\dfrac{4u^2}{g^2} - \dfrac{3u^2}{g^2}\right)}}{2} = \dfrac{u}{g} \pm \dfrac{u}{2g}.$

Taking negative sign, $t_1 = \dfrac{u}{2g}$.

Taking only positive sign, $t_2 = \dfrac{3u}{2g}$.

$$\therefore \qquad \frac{t_1}{t_2} = \frac{(u/2g)}{(3u/2g)} = \mathbf{1 : 3.}$$

15. Two particles, 1 and 2, move with constant velocities v_1 and v_2 along two mutually perpendicular straight lines towards the intersection point *O*. At the moment $t = 0$ the particles were located at the distances l_1 and l_2 from the point *O*. How soon will the distance the particles become the smallest ? What is it equal to ?



Soln. At a time *t*, the positions of the particle 1 and 2 are separately, $(l_1 - v_1 t, \theta)$ and $(0, l_2 - v_2 t)$. Hence the distance between them at this instant.

$$Z = \sqrt{(l_1 - v_1t)^2 + (l_2 - v_2t)^2}$$

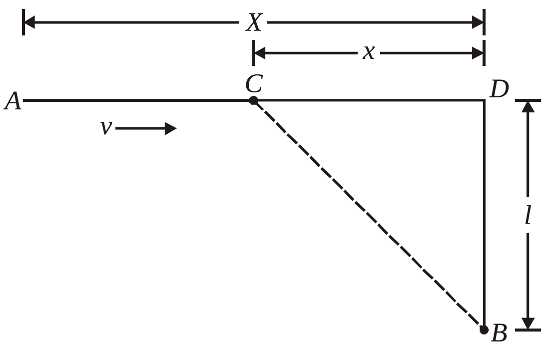
To be minimum at some instant $t = t'$, we use the condition

$$\frac{dZ}{dt} = 0 \Rightarrow \frac{1}{2}\frac{2(l_1 - v_1t)(-v_1) + 2(l_2 - v_2t)(-v_2)}{\sqrt{(l_1 - v_1t)^2 + (l_2 - v_2t)^2}} = 0$$

or $\quad (v_1^2 + v_2^2)t = l_1 v_1 + l_2 v_2 \quad$ or $\quad t = t' = \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$

Hence $\quad Z_{\min} = \sqrt{\left(l_1 - v_1\dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)^2 + \left(l_2 - v_2\dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)^2} = \dfrac{|l_1 v_2 - l_2 v_1|}{\sqrt{(v_1^2 + v_2^2)}}.$

16. From point *A* located on a highway, one has to get by car as soon as possible to point *B* located in the field at a distance *l* from the highway. It is known that the car moves in the field η times slower than on the highway. At what distance from the point *D* one must turn off the highway ?



Soln. Let the car turn off the road at a distance *x* from *D*. Hence the total time of travel by car from *A* to *B*.

$$T = \frac{X - x}{v} + \frac{\sqrt{l^2 + x^2}}{v/\eta}$$

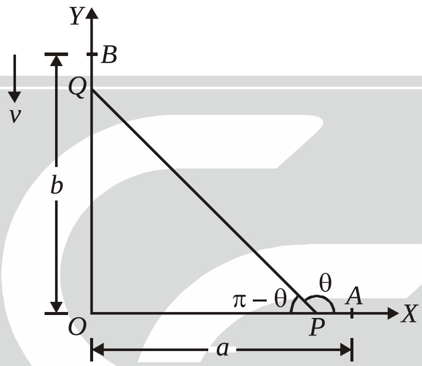
For this to be minimum :

$$\frac{dT}{dx}=0 \Rightarrow -\frac{1}{v}+\frac{\eta}{v}\frac{1}{2}\left(\frac{2x}{\sqrt{l^2+x^2}}\right)=0 \Rightarrow \frac{\eta x}{\sqrt{l^2+x^2}}=1$$

or $$x=\frac{\sqrt{l^2+x^2}}{\eta} \Rightarrow x^2\left(1-\frac{1}{\eta^2}\right)=\frac{l^2}{\eta^2} \Rightarrow x=\frac{l}{\sqrt{\eta^2-1}}.$$

17. Two points are moving with uniform velocities u and v along the perpendicular axes, OX and OY. The motion is directed towards O, the origin. When $t=0$, they are at a distance a and b respectively from O. Calculate the angular velocity of the line joining them at time t. Show that it is greatest, when $t=\frac{au+bv}{u^2+v^2}$.

Soln. See figure. At $t=0$, the particles are at A and B respectively.



Let at time t, the particles are at P and Q respectively.

From figure, $OP=a-ut$ ... (i)

and $OQ=b-vt$ ... (ii)

Let θ be the angle which the line PQ makes with the direction OX at time t. From ΔQOP,

$$\tan(\pi-\theta)=\frac{OQ}{OP}=\frac{b-vt}{a-ut} \text{ or } -\tan\theta=\frac{b-vt}{a-ut}$$

or $$\tan\theta=\left(\frac{b-vt}{ut-a}\right) \text{ or } \theta=\tan^{-1}\left(\frac{b-vt}{ut-a}\right) \quad \text{... (iii)}$$

Differentiating equation (iii), we get $$\frac{d\theta}{dt}=\frac{1}{1+\left(\frac{b-vt}{ut-a}\right)^2}\times\frac{(ut-a)(-v)-(b-vt)u}{(ut-a)^2}$$

Simiplifying it, we get $\omega=\frac{d\theta}{dt}=\frac{av-bu}{(a-ut)^2+(b-vt)^2}$, ω will be maximum when denominator is minimum because numerator is constant.

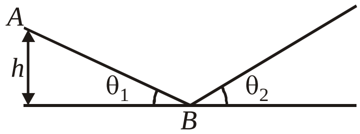
Let $$Z=(a-ut)^2+(b-vt)^2$$

$$\frac{dZ}{dt}=2(a-ut)(-u)+2(b-vt)(-v)=0$$

Solving it for t, we get $t=\frac{au+bv}{u^2+v^2}$.

It can be seen that $\frac{d^2Z}{dt^2}$ is positive. Hence ω is maximum, when $t = \frac{au+bv}{u^2+v^2}$.

18. A particle of mass m starts from rest moving down the inclined plane AB and rises up to point C on the inclined plane BC. Assuming no losses of energy, the time period of the to and for motion of the particle is............



Soln. The time taken t_1 in moving from A to B, is given by $\frac{h}{\sin\theta_1} = 0 + \frac{1}{2}(g\sin\theta_1)$ $(\because s = AB = \frac{h}{\sin\theta_1})$.

Therefore, $t_1 = \sqrt{\left(\frac{2h}{g}\right)\cdot\frac{1}{\sin\theta_1}}$

Similarly, the time taken, t_2 from C to B is given by $t_2 = \sqrt{\left(\frac{2h}{g}\right)\cdot\frac{1}{\sin\theta_2}}$.

Time period, $T = 2(t_1+t_2) = 2\sqrt{\left(\frac{2h}{g}\right)}\left[\frac{1}{\sin\theta_1}+\frac{1}{\sin\theta_2}\right]$.

19. A particle moves in the plane xy with constant acceleration a directed along the negative y-axis. The equation of motion of the particle has the form $y = px - qx^2$ where p and q are positive constants. Find the velocity of the particle at the origin of coordinates.

Soln. Given that $y = px - qx^2$

$$\therefore \quad \frac{dy}{dt} = p\frac{dx}{dt} - q\cdot 2x\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2} = p\frac{d^2x}{dt^2} - 2qx\frac{d^2x}{dt^2} - 2q\left(\frac{dx}{dt}\right)^2 \text{ or } -a = -2q\left(\frac{dx}{dt}\right)^2 = -2qv_x^2$$

$$\because \quad \frac{d^2x}{dt^2} = 0 \text{ (no acceleration along } x\text{-axis), and } \frac{d^2y}{dt^2} = -a.$$

$$\therefore \quad v_x^2 = \frac{a}{2q} \text{ or } v_x = \sqrt{\frac{a}{2q}}$$

Further, $\left(\frac{dy}{dt}\right)_{x=0} = p\frac{dx}{dt}$ or $v_y = pv_x$

$$\therefore \quad v_y = p\sqrt{\left(\frac{a}{2q}\right)}$$

Now, $v = \sqrt{(v_x^2+v_y^2)} = \sqrt{\left(\frac{a}{2q}+\frac{ap^2}{2q}\right)}$ or $v = \sqrt{\left[\frac{a(p^2+1)}{2q}\right]}$.