# Chapter 4

# Electrostatic Energy

## Electrostatic Energy:

The work necessary to assemble a system of charges against coulomb forces is stored in the field as potential energy. This is known as electrostatic energy.

### Electrostatic energy system of point charges:

Let at any time are place a point charge $q_1$ at $\vec{r}_1$. To place the charge $q_1$ at the position $\vec{r}_1$, we require no work because there is no interacting coulomb field.

$\therefore \qquad U_1 = 0$

But to bring the charge $q_2$ to the position $\vec{r}_2$ we require work done against coulomb repulsion due to $q_1$.

This work $\qquad U_2 = -\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{\infty}^{r_{12}} \frac{q_1 q_2}{r^2}\, dr = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|r_1 - r_2|} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$

Therefore for two point charges,

$$\boxed{u = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}}$$

To bring another charge, we need work done.

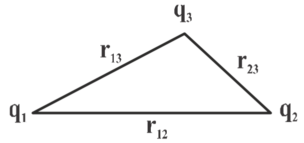
$$U_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1\mu_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Therefore, net work done

$$U = U_1 + U_2 + U_3 = 0 + \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

For N point charged,

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}}$$




We know electrostatic potential

$$\phi_i = \sum_{\substack{j=1 \\ i\neq j}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}$$

$\therefore \quad \boxed{U = \frac{1}{2}\sum_{i=1}^{N} q_i \phi_i}$

Potential energy of interaction of a charge 'q' with other charges

$u = qV$ , where $V$ is potential at location of $q$ due to other charges.

**For continuous charge distribution :**

For continous charge distribution u is given by

$\boxed{U = \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})\,dr}$ [ $\rho(\vec{r})$ is volume charge distribution]

## Electrostatic Energy in terms of field :

Suppose we have a finite region of space V in a dielectric medium of permitivity $\varepsilon$ and the volume charge density $\rho$ .

Therefore, the electrostatic energy of the system is given by

$$U = \frac{1}{2}\varepsilon_0 \int \rho(r)\phi(r)\,dV = \frac{1}{2}\int (\vec{\nabla}\cdot\vec{E})\phi\, dV \qquad \text{[Gauss's law } \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho \text{ ]}$$

$$= \frac{1}{2}\varepsilon_0 \int \left( (\vec{\nabla}\cdot\phi\vec{E}) - \vec{\nabla}\phi\cdot\vec{E} \right) dV$$

$$= \frac{1}{2}\varepsilon_0 \int (\vec{\nabla}\cdot\phi\vec{E})\,dV - \frac{1}{2}\int (\vec{\nabla}\phi\cdot\vec{E})\,dV$$

$$= \frac{1}{2}\varepsilon_0 \int (\phi\vec{E})\,dS + \frac{1}{2}(\vec{E}\cdot\vec{E})\,dV$$

At large distance from the charge distribution the first term will be vanish.

$\therefore \quad \boxed{U = \frac{1}{2}\varepsilon_0 \int E^2 dV}$

Therefore, the energy density, $\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$

## Forces and Torques from electrostatic energy:

We know work done = force . displacement

$\Rightarrow \quad dW = \vec{F}\cdot d\vec{r}$

Also for isolated system, the work done

$$dW = -dU = -\vec{\nabla}U\cdot d\vec{r}$$

$\therefore \quad \boxed{\vec{F} = -\vec{\nabla}U}$ , $u = -\int \vec{F}\cdot d\vec{r}$

If the part of the system under consideration is constrained to rotate about an axis and it undergoes a small rotation do under the action of a torque $\vec{\tau}$ then,

$$dW = \vec{\tau}\cdot d\vec{\theta} = -dU = -\nabla_\theta U\cdot d\theta$$

$\therefore \quad \boxed{\vec{\tau} = -\nabla_\theta U}$ and $u = \int \vec{F}\cdot d\vec{r}$

**The electrostatic energy associated with a uniform spherical charge distribution of total charge $q$ and radius $R$.**

The electric field of this distribution is

$$E = E_1 = \frac{qr}{4\pi\varepsilon_0 R^3} \text{ for } r \le R$$

$$E = E_2 = \frac{q}{4\pi\varepsilon_0 R^2} \text{ for } r \ge R$$

Since, the field is radially symmetric, the volume element may be expressed as $dv = 4\pi r^2 dr$, so that the energy is

$$U = \frac{\varepsilon_0}{2}\int E^2 dv = \frac{\varepsilon_0}{2}\int_0^R E_1^2 4\pi r^2 dr + \frac{\varepsilon_0}{2}\int_R^\infty E_2^2 4\pi r^2 dr$$

$$= \frac{1}{2}\int_0^R \frac{q^2 r^2}{16\pi^2\varepsilon_0 R^6} 4\pi r^2 dr + \frac{1}{2}\int_R^\infty \frac{q^2}{16\pi^2\varepsilon_0 r^4} 4\pi r^2 dr$$

$$= \frac{q^2}{40\pi\varepsilon_0 R} + \frac{q^2}{8\pi\varepsilon_0 R}, \quad \boxed{U = \frac{3q^2}{20\pi\varepsilon_0 R}}$$

**The electrostatic energy of a uniformly charged spherical shell of total charge $q$ and radius $a$.**

The electric field of the shell is

$$E = 0, \qquad \text{for } r \le R$$

$$E = \frac{q}{4\pi e_0 r^2} \qquad \text{for } r \ge R$$

The energy is therefore,

$$U = \frac{\varepsilon_0}{2}\int E^2 dv = \frac{1}{2}\int_R^\infty \frac{q^2}{16\pi^2\varepsilon_0 r^4} 4\pi r^2 dr$$

$$U = \frac{q^2}{8\pi\varepsilon_0 R}$$

**Example:** Consider a spherical charge distribution with a volume density

$$\rho(r) = \begin{cases} \rho_0 \cdot \dfrac{a}{r} & \text{for } 0 < r < a \\ 0 & \text{for } r > a \end{cases}$$

Show that the total electrostatic energy of the system is $\dfrac{Q^2}{6\pi \in_0 a}$, where Q is the total charge in the system.

**Soln.** Total charge in the system is

$$Q = \int_0^a \rho(r) \cdot 4\pi r^2 dr = 4\pi\rho_0 a \int_0^a r dr = 2\pi\rho_0 a^3$$

The electric field at any outside point can be obtained from Gauss's law as

$$E = \frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2} \text{ or } Q_{enclosed} = \int \rho d\tau = 4\pi\rho_0 a \int_0^a r dr = 2\pi\rho_0 a^3$$

For an internal point,

$$\int \vec{E}_i \cdot d\vec{S} = \frac{Q_r}{\epsilon_0} = \frac{1}{\epsilon_0}\int_0^r \rho(r)\cdot 4\pi r^2 dr = \frac{2\pi\rho_0 a r^2}{\epsilon_0}$$

$$\therefore \qquad E_i = \frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2} \text{ or } E_i = \frac{Q}{4\pi\epsilon_0 a^2}$$

For external point, $E_{out} = \frac{Q_{enclosed}}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$

Now, the total energy in the system is

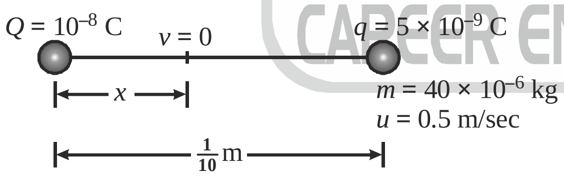
$$U = \int_0^a \frac{1}{2}\epsilon_0 E_i^2 \cdot 4\pi r^2 dr + \int_a^\infty \frac{1}{2}\epsilon_0 E_{out}^2 \cdot 4\pi r^2 dr$$

$$= 2\pi\epsilon_0 \int_0^a \left(\frac{Q}{4\pi\epsilon_0 a^2}\right)^2 r^2 dr + 2\pi\epsilon_0 \int_a^\infty \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \cdot \frac{dr}{r^2} = \frac{Q^2}{6\pi\epsilon_0 a}$$

## ADDITIONAL SOLVED PROBLEMS

1. A particle of mass 40 *mg* and carrying a charge $5 \times 10^{-9}$ C is moving directly towards a fixed positive point charge of magnitude $10^{-8}$ C. When it is at a distance of 10 cm from the fixed positive point charge it has a velocity of 50 cm sec$^{-1}$. At what distance from the fixed point charge will the particle come momentarily to rest? Is the acceleration constant during motion ?

**Soln.** Let $x$ be the distance of q from $Q$ when it comes to rest as shown in figure:




Use conservation of energy $(\text{K.E.} + \text{P.E.})_{\text{initial}} = (\text{K.E.} + \text{P.E.})_{\text{final}}$

$$\frac{1}{2}\times 40\times 10^{-6}\times(0.5)^2 + \frac{10^{-8}\times 5\times 9\times 10^9}{\left(\frac{1}{10}\right)} = 0 + \frac{10^{-8}\times 5\times 10^{-9}\times 9\times 10^9}{x}$$

Solving we get, $r = 4.737\times 10^{-2}$ m.

The force obeys inverse square law i.e., it increases as distance decreases. Hence, acceleration also increases.

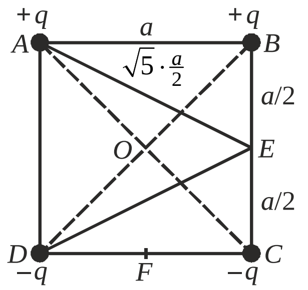
2. Four charges $+q, +q, -q$ and $-q$ are placed respectively at the corners $A$, $BC$ and $D$ of a square of side $a$, arranged in the given order. If $E$ and $F$ are mid points of side $BC$ and $CD$ respectively, what will be workdone in carrying a charge '$e$' from $O$ to $E$ and from $O$ to $F$.

**Soln.** The charge configuration is shown in figure :

Workdone in carrying a charge = change in potential energy = $q(V_{final} - V_{initial})$

$$AE^2 = a^2 + \left(\frac{a}{2}\right)^2 = \frac{5a^2}{4} \textbf{ or } AE = \frac{\sqrt{5}\cdot a}{2}$$

Potential at $E$ i.e., $V_E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{5}\cdot\frac{a}{2}} - \frac{q}{\sqrt{5}\cdot\frac{a}{2}} + \frac{q}{a/2} - \frac{q}{a/2}\right] = 0$




Potential at $F$ i.e., $V_F$ is given by

$$V_F = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{5}\cdot\frac{a}{2}} + \frac{q}{\sqrt{5}\cdot\frac{a}{2}} - \frac{q}{a/2} - \frac{q}{a/2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{4q}{\sqrt{5}\cdot a} - \frac{4q}{a}\right]$$

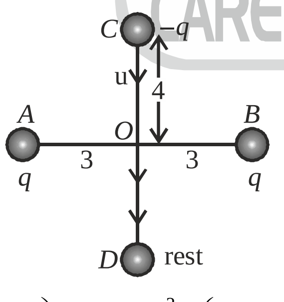
$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{4q}{a}\left[\frac{1}{\sqrt{5}} - 1\right]$$

Now, $W_{OE} = e(V_E - V_O) = e(0-0) = 0$ and $W_{OF} = e(V_F - V_O)$

$$= e\cdot\left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{4q}{a}\left\{\frac{1}{\sqrt{5}} - 1\right\} - 0\right] = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4eq}{a}\cdot\left[\frac{1}{\sqrt{5}} - 1\right].$$

3. Two fixed, equal positive charges, each of magnitude $5 \times 10^{-5}$ coul, are located at points $A$ and $B$ separated at a distance of 6 m. An equal and opposite charge moves towards them along the line $COD$, the perpendicular bisector of the line $AB$. The moving charge, when reaches the point $C$ at a distance of 4 m from $O$, has a kinetic energy of 4 joules. Calculate the distance of the farthest point $D$ at which the negative charge will reach before returning towards $C$.

**Soln.** Applying conservation of energy, $(K.E. + P.E.)_{at\ C} = (K.E. + P.E.)_{at\ D}$




$$\frac{1}{2}mu^2 - \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{AC} + \frac{1}{BC}\right) = 0 - \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{AD} + \frac{1}{BD}\right)$$

Put $q = 3\times10^{-5}c$

$\therefore \quad \frac{1}{2}mu^2 = 44J$

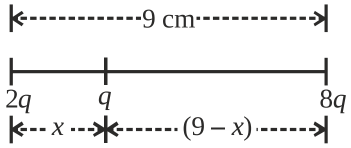
$AC = BC = \sqrt{3^2 + 4^2} = 5$ and solve to get

AD = BD = 9(m)

$\therefore \quad OD = \sqrt{(9)^2 - 3^2} = 6\sqrt{2}\ m$

4. Three charges, $q$, $2q$ and $8q$ are to be placed on 9 cm long straight line. Find the positions where the charges should be placed such that the potential energy of this system is minimum.

**Soln.** For minimum interaction energy, the charges of greater magnitudes should be placed at the extreme ends. The situation is shown in figure. Let the distance of charge $2q$ from charge $q$ be $x$ cm.




The potential energy of the system is given by

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q\times q}{x} + \frac{q\times 8q}{(9-x)} + \frac{2q\times 8q}{9}\right]$$

For minimum potential energy, $\partial U/\partial x = 0$

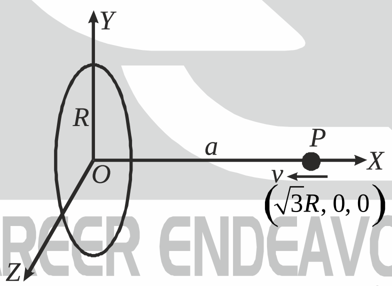
$$\therefore \quad \frac{\partial U}{\partial x} = \frac{1}{4\pi\varepsilon_0} \Rightarrow \frac{\partial}{\partial x}\left[\frac{2q^2}{x\times 10^{-2}} + \frac{8q^2}{(9-x)10^{-2}} + \frac{16q^2}{9\times 10^{-2}}\right] = 0$$

Solving we get $x = 3$ cm.

Thus the charge $q$ should be placed at a distance of 3 cm from the charge $2q$.

5. A circular ring of radius $R$ with uniform positive charge density $\lambda$ per unit length is located in the YZ-plane with its centre at the origin $O$. A particle of mass $m$ and positive charge $q$ is projected from the point $P\left(R\sqrt{3}, 0, 0\right)$ on the positive $X$-axis directly towards $O$, with initial velocity $v$. Find the smallest (non-zero) value of the speed $v$ such that the particle does not return to $P$.

**Soln.** The situation is shown in figure :




Particle will not return to P if it manages to at least reach center of ring. Potential at axial point of ring (charge $= Q$, radius $= R$) is $V_{ring} = \dfrac{Q}{4\pi\varepsilon_0\sqrt{R^2+x^2}}$

Potential energy of charge and ring = $q\,V_{ring}$

Appling conservation of energy

$$\left(K.E. + P.E.\right)_{x=\sqrt{3}\,R} = \left(K.E. + P.E.\right)_{x=0}$$

$$\frac{1}{2}mv^2 + \frac{qQ}{4\pi\varepsilon_0\sqrt{R^2+\left(\sqrt{3}\,R\right)^2}} = 0 + \frac{qQ}{4\pi\varepsilon_0\sqrt{R^2+0^2}} \quad \Rightarrow \quad v = \sqrt{\frac{qQ}{4\pi\varepsilon_0 Rm}}, \text{ since } Q = \lambda 2\pi R$$

$$\therefore \quad \boxed{v = \frac{\lambda q}{2\varepsilon_0 m}}$$

6. A conducting sphere $S_1$, of radius $r$ is attached to an insulating handle. Another conducting sphere $S_2$ of radius $R$ is mounted as an insulating stand. $S_2$ is initially uncharged. $S_1$ is given a charge $Q$, brought into contact with $S_2$ and removed. $S_1$ is recharged such that the charge on it is again $Q$ and is again brought into contact with $S_2$ and removed. This procedure is repeated $n$ times.

(a) Find the electrostatic energy of $S_2$ after $n$ such contacts with $S_1$.

(b) What is the limiting value of this energy as $n \to \infty$.

**Soln.** (a) Let $Q_2$ be the charge on sphere $S_2$ after $S_1$ is removed first $Q_1$ be charge on $S_1$.

$$\therefore \qquad Q_1 + Q_2 = Q$$

Since the potential at the surface of each sphere is the same, we have

$$\frac{Q_2}{4\pi\varepsilon_0 R} = \frac{Q_1}{4\pi\varepsilon_0 r} \Rightarrow \frac{Q_2}{R} = \frac{Q-Q_2}{r} \; i.e. \; Q_2 = \frac{1}{1+\frac{r}{R}} \text{ times, the total charge } Q_2 = \frac{Q}{1+\left(\frac{r}{R}\right)}$$

Now $S_1$ is recharged such that the charge on it is again $Q$. Charge on $S_2$ is $\frac{Q}{\left(1+\frac{r}{R}\right)}$. They are brought in contact.

Total charge on $S_1$ and $S_2 = \left[Q + \left\{\frac{Q}{\left(1+\frac{r}{R}\right)}\right\}\right]$

Therefore, now total charge on $Q_2$ will be

$$= \frac{1}{1+\frac{r}{R}} \text{ time total charge}$$

$$= \frac{1}{\left(1+\frac{r}{R}\right)} \cdot \left[Q + \frac{Q}{1+\frac{r}{R}}\right] = \frac{Q}{1+\frac{r}{R}} + \frac{Q}{\left(1+\frac{r}{R}\right)^2}$$

Similarly, after $n$ repeatation, total charge on $S_2$ becomes,

$$= \frac{Q}{\left\{1+\left(\frac{r}{R}\right)\right\}} + \frac{Q}{\left\{1+\left(\frac{r}{R}\right)\right\}^2} + \cdots + \frac{Q}{\left\{1+\left(\frac{r}{R}\right)\right\}^n}$$

$$= \frac{Q}{1+\left(\frac{r}{R}\right)} \left[\frac{1-1/\left(1+\frac{r}{R}\right)^n}{1-1/\left(1+\frac{r}{R}\right)}\right]$$

Electrostatic energy of $S_2$

$$= \frac{Q_{\text{final}}^2}{8\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R\left(1+\frac{r}{R}\right)^2} \times \left[\frac{1-1/\left(1+\frac{r}{R}\right)^n}{1-1/\left(1+\frac{r}{R}\right)}\right]^2$$

(b) Limiting energy at $n \to \infty$

$$E = \frac{Q^2}{8\pi\varepsilon_0 R\left(1+\frac{r}{R}\right)^2}\left[\frac{1-0}{1-1/\left(1+\frac{r}{R}\right)}\right]^2$$

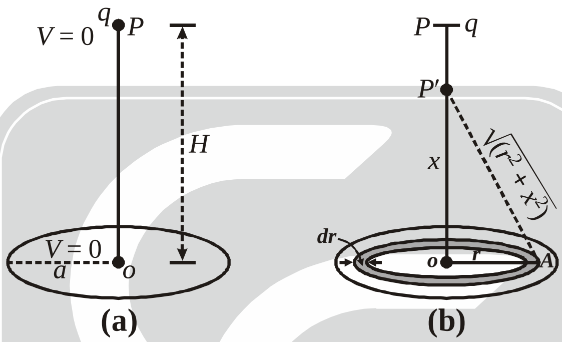
$$= \frac{Q^2}{8\pi\varepsilon_0 R}\left(\frac{R}{r}\right)^2 = \frac{Q^2 R}{8\pi\varepsilon_0 r^2}.$$

7. A non-conducting disc of radius $a$ and uniform surface charge density $\sigma$ is placed on the ground, with its axis vertical. A particle of mass $m$ and positive charge $q$ is dropped, along the axis of the disc, from a height $H$ with zero initial velocity. The particle has $q/m = 4\pi\varepsilon_0 g/\sigma$.

(a) Find the value of $H$ if the particle just reaches the disc.

(b) Sketch the potential energy of the particle as a function of its height and find its equilibrium position.

**Soln.** See below figure.




We know that, $U = qV_{disc}$

$$= \frac{\sigma q}{2\varepsilon_0}\left[(a^2+x^2)^{1/2} - x\right] \qquad \text{... (1)}$$

(a) Applying the principle of conservation of energy at points P and O, we have

(K.E. + P.E.)$_{\text{at P}}$ = (K.E. + P.E.)$_{\text{at O}}$

$$0 + mgH + \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{(a^2+H^2)} - H\right] = 0 + \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{(a^2+0^2)} - 0\right]$$

Given that $q = (4\varepsilon_0 g m/\sigma)$

$$\therefore \quad mgH + \frac{\sigma(4\varepsilon_0 g m/\sigma)}{2\varepsilon_0}\left[\sqrt{(a^2+H^2)} - H\right] = \frac{\sigma(4\pi\varepsilon_0 g m/\sigma)}{2\varepsilon_0} a$$

$$\text{or} \quad mgH + 2mg\left[\sqrt{(a^2+H^2)} - H\right] = 2mga$$

$$H + 2\sqrt{(a^2+H^2)} - 2H = 2a$$

$$2\sqrt{(a^2+H^2)} = (2a+H)$$

$$\text{or} \quad 4(a^2+H^2) = (2a+H)^2$$

Solving we get $H = (4a/3)$

(b) Total potential energy at a height $x$ above centre

$$\text{P.E.} = \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{(a^2+x^2)} - x\right] + mgx$$

Substituting the value of $q$, we get

$$\text{P.E.} = 2mg\left[\sqrt{(a^2+x^2)} - x\right] + m\,g\,x$$

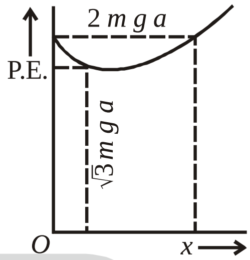
$$U = m\,g\left[2\sqrt{(a^2+x^2)} - x\right]$$

For equilibrium position, $\dfrac{dU}{dx} = 0$

$$\therefore \quad \frac{dU}{dx} = 0 = mg\left[\frac{2.2x}{2\sqrt{(a^2+x^2)}} - 1\right]$$

or $\quad \dfrac{2x}{\sqrt{(a^2+x^2)}} = 1 \quad$ or $\quad 4x^2 = (a^2+x^2)$

$$\therefore \quad x = \frac{a}{\sqrt{3}}$$




The variation of potential energy with $x$ is shown in figure.