Chapter 5

ISOMORPHISM

1. Isomorphic Mapping. Definition: Suppose G and G' are two groups, the composition in each being denoted multiplicatively. A mapping *f* of *G* into G' is said to be an isomorphic mapping of *G* into G' if

(i) *f* is one-to-one i.e., distinct elements in *G* have distinct *f*-images in G',

(ii) $f(ab) = f(a)f(b) \forall a, b \in G$ i.e., the image of the product is the product of the images.

It should be noted that when we say that *f* is a mapping of *G* into G', we usually include in it the possibility that the mapping *f* may be onto G'. If an isomorphic mapping *f* of *G* into G' is onto G', then it is called an isomorphic mapping of *G* onto G'.

If *f* is an isomorphic mapping of a group *G* into a group G', then *f* is also called an isomorphism of *G* into G'. If *f* is an isomorphism of *G* onto G', the group G' is called an isomorphic image of the group *G*. Also then we say that the group G is isomorphic to the group G'. Thus we can give the complete definition of isomorphic groups like this:

2. Isomorphic groups. Definition: Suppose *G* and G' are two groups. Further suppose that the compositions in both *G* and G' have been denoted multiplicatively. Then we say that the group G is isomorphic to the group G' if there exists a one-to-one mapping *f* of *G* onto G' such that

$f(ab) = f(a)f(b) \forall a, b \in G$ i.e., the mapping *f* preserves the compositions in *G* and G'.

If the group *G* is isomorphic to the group G', symbolically we write $G \cong G'$.

Note 1: If *G* is isomorphic to G', there may exist more than one isomorphisms of *G* onto G'.

Note 2: If the group *G* is finite, then *G* can be isomorphic to G' only if G' is also finite and the number of elements of *G* is equal to the number of elements in G'. Otherwise there will exist no mapping *f* from *G* to G' which is one-one as well as onto.

Note 3: If the group *G* is isomorphic to the group G', then we say that the groups *G* and G' are abstractly identical. From the point of view of abstract algebra we shall regard them as one group and not as two different groups.

3. Some more examples:

Example 1: If $\mathbb{R}$ is the additive group of real numbers and $\mathbb{R}^+$ the multiplicative group of positive real numbers, prove that the mapping $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x \ \forall x \in \mathbb{R}$ is an isomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$.

Soln. If *x* is any real number, positive, zero or negative, then e^x is always a positive real number. Also e^x is unique. Therefore if $f(x) = e^x$ then $f : \mathbb{R} \to \mathbb{R}^+$.

***f* is one-to-one.**

Let $x_1, x_2 \in \mathbb{R}$. Then $f(x_1) = f(x_2) \quad \Rightarrow e^{x_1} = e^{x_2}$

$\Rightarrow \log e^{x_1} = \log e^{x_2} \Rightarrow x_1 \log e = x_2 \log e \Rightarrow x_1 = x_2$

Thus, two elements in $\mathbb{R}$ have the same f-image in $\mathbb{R}^+$ only if they are equal. Consequently distinct elements in $\mathbb{R}$ have distinct f-images in $\mathbb{R}^+$. Therefore f is one-to-one.

f is onto: Suppose y is any element of $\mathbb{R}^+$ i.e. y is any positive real number. Then $\log y$ is a real number i.e., $\log y \in \mathbb{R}$.

Now $f(\log y) = e^{\log y} = y$. Therefore each element of $\mathbb{R}^+$ is the f-image of some element of $\mathbb{R}$. Thus f is onto.

f preserves compositions in $\mathbb{R}$ and $\mathbb{R}^+$. Suppose x_1 and x_2 are any two elements of $\mathbb{R}$. Then

$$f(x_1 + x_2) = e^{x_1 + x_2}$$

$$= e^{x_1} . e^{x_2}$$

$$= f(x_1) f(x_2) \; [\because f(x_1) = e^{x_1} \text{ and } f(x_2) = e^{x_2}]$$

Thus f preserves compositions in $\mathbb{R}$ and $\mathbb{R}^+$. Here the composition in R is addition and the composition in $\mathbb{R}^+$ is multiplication. Therefore f is an isomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$. Hence $\mathbb{R} \cong \mathbb{R}^+$.

Example 2. Let $\mathbb{R}^+$ be the multiplicative group of all positive real numbers and $\mathbb{R}$ be the additive group of all real numbers. Show that the mapping $g : \mathbb{R}^+ \to \mathbb{R}$ defined by

$g(x) = \log x \; \forall x \in \mathbb{R}^+$ is an isomorphism.

Soln. If x is any positive real number, then $\log x$ is definitely a real number. Also $\log x$ is unique. Therefore, if $g(x) = \log x$, then $g : \mathbb{R}^+ \to \mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}^+$. Let $g(x_1) = g(x_2)$

$\Rightarrow \log x_1 = \log x_2 \Rightarrow e^{\log x_1} = e^{\log x_2} \Rightarrow x_1 = x_2$

Therefore, g is one-to-one.

Suppose y is any element of $\mathbb{R}$ i.e. y is any real number. Then e^y is definitely a positive real number i.e. $e^y \in \mathbb{R}^+$.

Now $g(e^y) = \log e^y = y$. Thus $y \in \mathbb{R} \Rightarrow$ that there exists $e^y \in \mathbb{R}^+$ such that $g(e^y) = y$. Therefore each element of $\mathbb{R}$ is the g-image of some element of $\mathbb{R}^+$. Thus g is onto.

g preserves compositions in $\mathbb{R}^+$ and $\mathbb{R}$. Suppose x_1 and x_2 are any two elements of $\mathbb{R}^+$. Then

$$g(x_1 x_2) = \log(x_1 x_2) \text{ [by def. of } g]$$

$$= \log x_1 + \log x_2$$

$$= g(x_1) + g(x_2) \text{ [by def. of } g]$$

Thus g preserves compositions in $\mathbb{R}^+$ and $\mathbb{R}$. Here the composition in $\mathbb{R}^+$ is multiplication and the composition in $\mathbb{R}$ is addition. Therefore g is an isomorphism of $\mathbb{R}^+$ onto $\mathbb{R}$. Hence $\mathbb{R}^+ \cong \mathbb{R}$.

Ex.3. Show that the additive group of integers $G = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ is isomorphic to the additive group $G' = \{..., -3m, -2m, -1m, 0, 1m, 2m, 3m, ...\}$ where m is any fixed integer not equal to zero.

Soln. If $x \in G$, then obviously $mx \in G'$. Let $f : G \to G'$ be defined by $f(x) = mx \; \forall x \in G$.

Let $x_1, x_2 \in G$. Let $f(x_1) = f(x_2)$

$\Rightarrow mx_1 = mx_2$ [by def. of f]

$\Rightarrow x_1 = x_2$ $[\because m \neq 0]$

Therefore f is one-to-one

Suppose y is any element of G'. Then obviously $y/m \in G$. Also $f(y/m) = m(y/m) = y$.

Thus, if $y \in G'$ then there exists $y/m \in G$ such that $f(y/m) = y$. Therefore each element of G' is the f-image of some element of G. Hence f is onto.

Again, if x_1 and x_2 are any two elements of G, then

$f(x_1 + x_2) = m(x_1 + x_2)$ [by def. of f]

$= mx_1 + mx_2$ [by distributive law for integers]

$= f(x_1) + f(x_2)$ [by definition of f]

Thus, f preserves compositions in G and G'. Therefore, f is an isomorphic mapping of G onto G'. Hence, G is isomorphic to G'.

Ex.4. Show that the set $\mathbb{C}$ of all complex numbers under addition is a group which is isomorphic to itself under the identity mapping as well as under the mapping which takes every complex number into its conjugate complex.

Soln. The identity mapping f defined by $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z \; \forall z \in \mathbb{C}$ is obviously one-one onto.

Also, $f(z_1 + z_2) = z_1 + z_2 = f(z_1) + f(z_2) \forall z_1, z_2 \in \mathbb{C}$.

$\therefore$ the identity mapping f is an isomorphism of $\mathbb{C}$ onto $\mathbb{C}$.

If $z = x + iy$ is any complex number, then $\bar{z} = x - iy$ is called the conjugate complex of z.

Let $g : \mathbb{C} \to \mathbb{C}$ be such that $g(z) = \bar{z} \; \forall z \in \mathbb{C}$

Let $z_1, z_2 \in \mathbb{C}$. Then $g(z_1) = g(z_2) \Rightarrow \bar{z}_1 = \bar{z}_2 \Rightarrow (\bar{z}_1) = (\bar{z}_2) \Rightarrow z_1 = z_2$

Therefore, g is one-to-one.

If $x + iy$ is any element of $\mathbb{C}$, then $x - iy$ is also an element of $\mathbb{C}$. Also $g[(x - iy] = x + iy$. Therefore g is onto.

Further, if $z_1, z_2 \in \mathbb{C}$, then $g(z_1 + z_2) = \overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2 = g(z_1) + g(z_2)$

Hence g is also an isomorphism of $\mathbb{C}$ onto $\mathbb{C}$.

4. Some important properties of isomorphic mappings:

Let f be an isomorphic mapping of a group G into a group G'. Then we have the following important properties.

(i) The f-image of the identity e of G is the identity of G' i.e., $f(e)$ is the identity of G'.

Proof: Let e be the identity of G and e' be the identity of G'. Let a be any element of G. Then $f(a) \in G'$.

Now, $e'f(a) = f(a)$ [$\because$ e' is the identity of G]

$= f(ea)$ [$\because$ e is the identity of G]

$= f(e)f(a)$ [$\because$ f is an isomorphic mapping]

Now in the group G', we have

$e'f(a) = f(e)f(a) \Rightarrow e' = f(e)$ [by right cancellation law in G']

$\therefore$ $f(e)$ is the identity of G'.

(ii) The f-image of the inverse of an element a of G is the inverse of the f-image of a i.e., $f(a^{-1}) = [f(a)]^{-1}$

Proof : Suppose e is the identity of G and e' is the identity of G'. Then $f(e)=e'$. Now let a be any element of G. Then $a^{-1}\in G$ and $aa^{-1}=e$. We have

$e'=f(e)=f(aa^{-1})=f(a)f(a^{-1})$ [$\because f$ is composition preserving]

Therefore, $f(a^{-1})$ is the inverse of $f(a)$ in the group G'. Thus $f(a^{-1})=[f(a)]^{-1}$

(iii) The order of an element a of G is equal to the order of its image $f(a)$.

Proof : Suppose e is the identity of G. Then $f(e)$ is the identity of G'. Let the order of a be finite and let it be equal to n.

Then $a^n=e\Rightarrow f(a^n)=f(e)\Rightarrow f(aaa...n \text{ times})=f(e)$

$\Rightarrow f(a)f(a)...n \text{ times}=f(e)$

$\Rightarrow [f(a)]^n=f(e)\Rightarrow$ order of $f(a)\le n$

If now the order of $f(a)$ is m, then

$[f(a)]^m=f(e)\Rightarrow f(a)f(a)f(a)...m \text{ times}=f(e)$

$\Rightarrow f(aaa...m \text{ times})=f(e)\Rightarrow f(a^m)=f(e)$

$\Rightarrow a^m=e$ [$\because f$ is one-one]

$\Rightarrow$ order of $a\le m$

Thus, $m\le n$ and $n\le m\Rightarrow m=n$

If the order of a is infinite, then the order of $f(a)$ cannot be finite. Because if the order of $f(a)$ is finite and is equal to m, then we have $a^m=e$. Therefore the order of a is finite. Thus we get a contradiction.

5. The relation of isomorphism in the set of all groups.

Theorem: The relation of isomorphism in the set of all groups is an equivalence relation.

Proof: We shall prove that the relation of isomorphism denoted by $\cong$ in the set of all groups is reflexive, symmetric and transitive.

Reflexive: If G is any group, then $G\cong G$. Let f be the identity mapping on G i.e., let $f:G\to G$ such that $f(x)=x,\forall x\in G$. Obviously f is one-one onto. Also if x, y are any elements of G, then $f(x)=x$ and $f(y)=y$.

Also, $f(xy)=xy$ [$\because f$ is identity mapping]

$=f(x)f(y)$

$\therefore f$ is composition preserving also. Thus f is an isomorphism on G onto G.

Hence, $G\cong G$

Symmetric [i.e. $G\cong G'\Rightarrow G'\cong G$]. Suppose a group G is isomorphic to another group G'. Let f be an isomorphism of G onto G'. Then f is one-one onto and preserves compositions in G and G'. Since f is one-one onto, therefore, it is invertible i.e. f^{-1} exists. Also we know that the mapping f^{-1} is also one one onto.

Now we shall show that $f^{-1}:G'\to G$ is also composition preserving. Let x',y' be any elements of G'. Then there exist elements $x,\ y\in G$ such that

$f^{-1}(x')=x,\ f^{-1}(y')=y$ ...(1)

and $f(x)=x',\ f(y)=y'$ ...(2)

Now , $f^{-1}(x'\ y') = f^{-1}[f(x)f(y)]$ [From (2)]

$= f^{-1}[f(xy)]$, since $f(xy) = f(x)f(y)$

$= xy$ [by def. of f^{-1}]

$= f^{-1}(x')f^{-1}(y')$ [From (1)]

$\therefore$ f^{-1} preserves compositions in G' and G.

Hence, $G' \cong G$.

Transitive [*i.e.*, $G \cong G', G' \cong G'' \Rightarrow G \cong G''$]: Suppose G is isomorphic to G' and G' is isomorphic to G''. Further, suppose that $f : G \to G'$ and $g : G' \to G''$ are the respective isomorphic mappings. **We** know that the composite mapping $g\,o\,f : G \to G''$ defined by

$(gof)(x) = g[f(x)]; \forall x \in G$

is also one-one onto if both f and g are one-one onto.

Further, if x, y are any elements of G, then

$(gof)(xy) = g[f(xy)]$ [by definition of gof]

$= g[f(x)f(y)]$ [$\because$ f is composition preserving]

$= g[f(x)]g[f(y)]$ [$\because$ g is also an isomorphism]

$= [(gof)(x)]\,[(gof)(y)]$

Hence, gof preserves compositions in G and G''.

$\therefore$ gof is an isomorphism of G onto G'' and $G \cong G''$.

Hence, the relation of isomorphism in the set of all groups is an equivalence relation.

Note: The relation of isomorphism in the set of all groups will partition the set of all groups into disjoint equivalence classes. If G_1 is any group, then all the groups isomorphic to G_1 will form one equivalence class. If G_2 is another group not isomorphic to G_1, then all the groups isomorphic to G_2 will form another equivalence class, and so on.

Example: Any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$. Any infinite cyclic group is isomorphic to $\mathbb{Z}$.

Example: The mapping from $\mathbb{R}$ under addition to itself given by $\phi(x) = x^3$ is not an isomorphism. Although, ϕ is one-to-one and onto, it is not operation-preserving since it is not true that $(x+y)^3 = x^3 + y^3$ for x and y.

Example: $U(10) \not\approx U(12)$. This is a bit trickier to prove. First, note that $x^2 = 1$ for all x in $U(12)$. Now, suppose that ϕ is an isomorphism from $U(10)$ onto $U(12)$. Then,

$\phi(9) = \phi(3.3) = \phi(3)\phi(3) = 1$ and $\phi(1) = \phi(1.1) = \phi(1)\phi(1) = 1$

Thus, $\phi(9) = \phi(1)$, but $9 \neq 1$, which is a contradiction to the supposed one-to-one character of ϕ.

Example: There is no isomorphism from $\mathbb{Q}$, the group of rational numbers under addition, to $\mathbb{Q}^*$, the group of nonzero rational numbers under multiplication. If ϕ were such a mapping, there would be a rational number a such that $\phi(a) = -1$. But then

$$-1 = \phi(a) = \phi\left(\frac{1}{2}a + \frac{1}{2}a\right) = \phi\left(\frac{1}{2}a\right)\phi\left(\frac{1}{2}a\right) = \left[\phi\left(\frac{1}{2}a\right)\right]^2$$

However, no rational number squared is –1.

Example: Let $G = SL(2, \mathbb{R})$, the group of 2 × 2 real matrices with determinant 1. Let M be any 2×2 real matrix with determinant 1. Then we can define a mapping from G to G itself by $\phi_M(A) = MAM^{-1}$ for all A in G. To verify that ϕ_M is an isomorphism we carry out the four steps.

Step 1: ϕ_M is a function from G to G. Here, we must show that $\phi_M(A)$ is indeed an element of G whenever, A is an element of G. This follows from properties of determinants:
$\det(MAM^{-1}) = (\det M)(\det A)(\det M)^{-1} = 1.1.1^{-1} = 1.$
Thus, MAM^{-1} is in G.

Step-2: ϕ_M is one-to-one. Suppose that $\phi_M(A) = \phi_M(B)$. Then $MAM^{-1} = MBM^{-1}$ and, by left and right cancellation, A = B.

Step-3: ϕ_M is onto. Let B belong to G. We must find a matrix A in G such that $\phi_M(A) = B$. If such a matrix A is to exist, it must have the property that $MAM^{-1} = B$. But this tells us exactly what A must be! For we can solve for A to obtain $A = M^{-1}BM$.

Step-4: ϕ_M is operation preserving. Let A and B belong to G. Then,

$\phi_M(AB) = M(AB)M^{-1} = MA(M^{-1}M)BM^{-1}$

$= (MAM^{-1})(MBM^{-1}) = \phi_M(A)\phi_M(B)$

The mapping ϕ_M is called conjugation by M.

6. Theorem: Cayley's Theorem: Every group is isomorphic to a group of permutation.

Proof: To prove this, let G be any group. We must find a group G' of permutations that is isomorphic to G. Since G is all we have to work with, we will have to use it to construct G'. For any g in G, define a function T_g from G to G by $T_g(x) = gx$ for all x in G.
(In words, T_g is just multiplication by g on the left). T_g is a permutation on the set of elements of G. Now, let $G' = \{T_g \mid g \in G\}$. Then, G' is a group under the operation of function composition. To verify this, we first observe that for any g and h in G we have $T_gT_h(x) = T_g(T_h(x)) = T_g(hx) = g(hx) = (gh)x = T_{gh}(x)$, so that $T_gT_h = T_{gh}$. From this it follows that T_e is the identity and $(T_g)' = T_{g'}$. Since function composition is associative, we have verified all the conditions for G' to be a group.

The isomorphism ϕ between G and G' is now ready made. For every g in G, define $\phi(g) = T_g$. Clearly, $g = h$ implies $T_g = T_h$, so that ϕ is a function from G to G. On the other hand, if $T_g = T_h$, then $T_g(e) = T_h(e)$ or $ge = he$. Thus, ϕ is one-to-one. By the way G' was constructed, we see that ϕ is onto. The only condition that remains to be checked is that ϕ is operation preserving. To this end let x and y belong to G. Then $\phi(xy) = T_{xy} = T_xT_y = \phi(x)\phi(y)$

The group G' constructed above is called left regular representation of G.

7. Theorem: Properties of Isomorphisms: Suppose that ϕ is an isomorphism from a group G onto a group G'. Then

1. ϕ carries the identity of G to the identity of G'.
2. For every integer n and for every group element a in G, $\phi(a^n) = [\phi(a)]^n$
3. For any elements a and b in G, a and b commute if and only if $\phi(a)$ and $\phi(b)$ commute.
4. G is Abelian if and only if G' is Abelian.
5. $|a| = |\phi(a)|$ for all a in G (isomorphisms preserve orders).

6. G is cyclic if and only if G' is cyclic.

7. For a fixed integer k and a fixed group element b in G, the equation $x^k = b$ has the same number of solutions in G as does the equation $\phi(x^k) = \phi(b)$ in G'.

8. ϕ' is an isomorphism from G' onto G.

9. If K is a subgroup of G, then $\phi(K) = \{\phi(k) \mid k \in K\}$ is a subgroup of G'.

Proof: We will restrict ourselves to proving only properties 1, 2, and 5. Note, however, that property 4 follows directly from property 3, and property 6 directly from property 5. For convenience, let us denote the identity in G by e_G, and the identity in G' by $e_{G'}$. Then $e_G = e_G e_G$ so that

$\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$

But $\phi(e_G) \in G'$, so that $\phi(e_G) = e_{G'}\phi(e_G)$, as well. Thus, by cancellation, we have $e_{G'} = \phi(e_G)$. This proves property 1.

For positive integers, property 2 follows from the definition of a homomorphism and mathematical induction. If n is negative, then-n is positive and we have from property 1 and the observation about the positive integer case that $e = \phi(e) = \phi(g^n g^{-n}) = \phi(g^n)\phi(g^{-n}) = \phi(g^n)(\phi(g))^{-n}$. Thus, multiplying both sides on the right by $(\phi(g))^n$, we have $(\phi(g))^n = \phi(g^n)$.

To prove property 5, we note that $a^n = e$ if and only if $\phi(a^n) = \phi(e)$. So, by properties 1 and 2, $a^n = e$ if and only if $(\phi(a))^n = e$. Thus, a has infinite order if and only if $\phi(a)$ has infinite order, and a has finite order n if and only if $\phi(a)$ has order n.

8. Definition: Automorphism: An isomorphism from a group G onto itself is called an automorphism of G.

Example: The function ϕ from $\mathbb{C}$ to $\mathbb{C}$ given by $\phi(a+bi) = a - bi$ is an automorphism of the group of complex numbers under addition. The restriction of ϕ to $\mathbb{C}^*$ is also an automorphism of the group of the nonzero complex numbers under multiplication.

Example: Let $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$. Then $\phi(a,b) = (b,a)$ is an automorphism of the group $\mathbb{R}^2$ under component-wise addition. Geometrically, ϕ reflects each point in the plane across the line $y = x$. More generally, any reflection across a line passing through the origin or any rotation of the plane about the origin is an automorphism of $\mathbb{R}^2$.

9. Definition: Inner Automorphism induced by *a*: Let G be a group, and let $a \in G$. The function ϕ_a defined by $\phi_a(x) = axa^{-1}$ for all x in G is called the inner automorphism of G induced by a.

10. Theorem: Aut(*G*) and Inn(*G*) Are Groups: The set of automorphism of a group and the set of inner automorphism of a group are both groups under the operation function composition.

11. Theorem: Aut($\mathbb{Z}_n$) $\approx$ *U*(*n*): For every positive integer n, Aut($\mathbb{Z}_n$) is isomorphic to $U(n)$.

SOLVED EXAMPLES

(i). Let Aut (G) denote the group of automorphism of a group G, which one of the following is not a cyclic group.

(a) Aut ($\mathbb{Z}_4$) (b) Aut ($\mathbb{Z}_6$) (c) Aut ($\mathbb{Z}_8$) (d) Aut ($\mathbb{Z}_{10}$)

[GATE-2009]

Soln. By theorem Aut $(\mathbb{Z}_n) \approx U(n)$

(a) Aut $(\mathbb{Z}_4) \approx U(4) = \{1, 2\} \Rightarrow$ cyclic

(b) Aut $(\mathbb{Z}_6) \approx U(6) = \{1, 5\} \Rightarrow$ cyclic

(c) Aut $(\mathbb{Z}_8) \approx U(8) = \{1, 3, 5, 7\} \Rightarrow$ not cyclic

Hence, correct option is (c).

(ii). Let G be a cyclic group of order 8, then its group of automorphism has order

(a) 2 (b) 4 (c) 6 (d) 8

[GATE-2006]

Soln. Let G be a cyclic group of order 8 then $G \approx \mathbb{Z}_8$ by formula Aut $(\mathbb{Z}_n) \approx U(n)$, so

Aut $(\mathbb{Z}_8) \approx U(8) = \{1, 3, 5, 7\}$

So order of group of automorphism is 4.

Hence, correct option is (b).

(iii). Let $(\mathbb{Z}, +)$ denote the group of all integer under addition. The number of all automorphism of $(\mathbb{Z}, +)$ is

(a) 1 (b) 2 (c) 3 (d) 4

[GATE-2001]

Soln. To find number of isomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{Z}, +)$ then $n \to n,\ n \to -n$ are only two isomorphism.

So number of all automorphism of $(\mathbb{Z}, +)$ are 2. infact Aut $(\mathbb{Z}) \approx \mathbb{Z}_2$

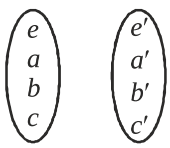
Hence, correct option is (b).

(iv). The order of the automorphism group of Klein's group is

(a) 3 (b) 4 (c) 6 (d) 24

[D.U. 2014]

Soln. Aut (K_4)



a has three choices as a', b', c' and once a maps to any one of these elements then for b or c there are two remaining choices.

If we choose b then c is already fixed as $c = a \cdot b$.

Hence, total maps are $3 \times 2 = 6$.

Hence, correct option is (c).

(v). Which one of the following group is cyclic?

(a) The group of positive rational numbers under multiplication

(b) The dihedral group of order 30

(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_{15}$

(d) Automorphism group of $\mathbb{Z}_{10}$

[D.U. 2014]

Soln. Aut$(\mathbb{Z}_{10}) \cong$ U(10) $\cong \mathbb{Z}_4$.

Hence, correct option is (d).

(vi). The logarithmic map from the multiplicative group of positive real numbers to the additive group of real number is

(a) a one-to-one but not an onto homomorphism (b) an onto but not a one-to-one homomorphism

(c) not a homomorphism (d) an isomorphism

[D.U. 2014]

Soln. An isomorphism

$\phi:(\mathbb{R}^+,\cdot)\to(\mathbb{R},+)$ defined by $\phi(a)=\log a$

let, $a,b\in\mathbb{R}^+$ then

$\phi(a\cdot b)=\log(ab)=\log a+\log b$

$=\phi(a)+\phi(b)$

Group homomorphism

Let $\phi(a)=0\Rightarrow\log a=0$

$\Rightarrow\boxed{a=1}$. Hence, identity maps to identity only $\Rightarrow\phi$ is 1 – 1.

again, let $b\in(\mathbb{R},+)$ then $\exists$ an element e^b such that $\phi(e^b)=\log e^b=b\cdot\log e=b$ (as $\log e=1$)

$\Rightarrow\phi$ is onto

Thus ϕ is an isomorphism.

Hence, correct option is (d).

(vii). If f is a group homomorphism from $(\mathbb{Z},+)$ to $(\mathbb{Q}-\{0\},\cdot)$ such that $f(2)=1/3$, then the value $f(-8)$ is

(a) 81 (b) 1/81 (c) 1/27 (d) 27

[D.U. 2014]

Soln. Given f is a group homomorphism from $f:(\mathbb{Z},+)\to(\mathbb{Q}-\{0\},\cdot)$ s.t. $f(2)=\frac{1}{3}$ then $f(-8)=?$

$f(2)=\frac{1}{3}\Rightarrow f(-2)=3$ [as inverse of $\frac{1}{3}$ in 3]

$\Rightarrow f(-8)=f(-2-2-2-2)=f(-2)\cdot f(-2)\cdot f(-2)\cdot f(-2)$

$=3\cdot3\cdot3\cdot3=81\cdot$

Hence, correct option is (a).

(viii). The quotient group $Q_8/\{1,-1\}$ is isomorphic to

(a) $(Q_8,\cdot)$ (b) $(\{1,-1\},\cdot)$ (c) $(V_4,+)$ (d) $(\mathbb{Z}_4,+)$

[D.U. 2014]

Soln. Since {1, –1} is centre of the group Q_8. Hence Q_8/{1, –1} is a well defined group as {1, –1} is normal subgroup of Q_8. And also since {1, –1} is not trivial.

$$\Rightarrow\frac{Q_8}{\{1,-1\}}\not\cong Q_8$$

again $\Rightarrow\frac{Q_8}{\{1,-1\}}\not\cong G$ where G is cyclic.

Since for any group if $\frac{G_1}{H} \cong G_2$ where G_2 is cyclic $\Rightarrow$ G_1 is abelian.

But since Q_8 is not an abelian, thus $\frac{Q_8}{\{1,-1\}}$ is not isomorphic to a cyclic group. Hence (b) and (d) are incorrect.

Hence, correct option is (c).

(ix). The quotient group $\mathbb{R}/\mathbb{Z}$ is

(a) an infinite Abelian group

(b) cyclic

(c) the same as $\{r+\mathbb{Z} : 0 \le r < 1\}$

(d) isomorphic to the multiplicative group of all complex numbers of unit modulus

[D.U. 2014]

Soln. Since quotient group of an abelian group is also abelian and also $\mathbb{R}/\mathbb{Z}$ is an infinite group isomorphic to the multiplicative group of all complex numbers of unit modulus.

$\mathbb{R}/\mathbb{Z} = \{r+\mathbb{Z} : 0 \le r < 1\}$.

Hence, correct option are (a), (c) and (d).

(x). Which of the following pairs of groups are isomorphic to each other?

(a) $\langle \mathbb{Z},+\rangle, \langle \mathbb{Q},+\rangle$ (b) $\langle \mathbb{Q},+\rangle, \langle \mathbb{R}^+,\cdot\rangle$

(c) $\langle \mathbb{R},+\rangle, \langle \mathbb{R}^+,\cdot\rangle$ (d) $\text{Aut}(\mathbb{Z}_3), \text{Aut}(\mathbb{Z}_4)$ **[D.U. 2014]**

Soln. $\langle \mathbb{R},+\rangle \cong \langle \mathbb{R}^+,\cdot\rangle$

$f : (\mathbb{R},+) \to (\mathbb{R}^+,\cdot)$ defined by $f(x) = e^x$

f is 1 – 1 onto homomorphism

Also, $Aut(\mathbb{Z}_3) \cong U(3) \cong \mathbb{Z}_2$

also $Aut(\mathbb{Z}_4) \cong U(4) \cong \mathbb{Z}_2$

Hence, correct option are (c) and (d).

(xi). The number of elements in the group $Aut\,\mathbb{Z}_{200}$ of all automorphisms of $\mathbb{Z}_{200}$ is

(a) 78 (b) 80 (c) 84 (d) 82 **[D.U. 2015]**

Soln. Aut $(Z_n) \cong U(n)$ and $|U(n)| = \phi(n)$

$\Rightarrow$ Aut $\mathbb{Z}_{200} = \phi(200)$

$= \phi\cdot(2^3 \times 5^2) = \phi(2^3)\cdot\phi(5^2)$

$= (2^3 - 2^2)\cdot(5^2 - 5) = 4 \times 20 = 80$

Hence, correct option is (b).

(xii). Let $G = U(32)$ and $H = \{1, 31\}$. The quotient group G/H is isomorphic to

(a) $\mathbb{Z}_8$ (b) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ (c) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (d) The dihedral group D_4

[D.U. 2015]

Soln. $U(32) \cong U(2^5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_8$$

And also, $H = \{1, 31\} \cong \mathbb{Z}_2$

So, $\dfrac{G}{H} = \dfrac{\mathbb{Z}_2 \oplus \mathbb{Z}_8}{\mathbb{Z}_2} \cong \mathbb{Z}_8$.

Hence, correct option is (a).

(xiii). For a group G. Let Aut (G) denote the group of automorphism of G. Which of the following statement is true?

(a) Aut $(\mathbb{Z})$ is isomorphic $\mathbb{Z}_2$
(b) if G is cyclic then Aut (G) is cyclic
(c) if Aut (G) is trivial then G is trivial
(d) Aut $(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$

[TIFR-2015]

Soln. Automorphism on $\mathbb{Z}$ are as follow $f : \mathbb{Z} \to \mathbb{Z}$; $f(n) = n$ and $\phi(n) = -n$

Aut $(\mathbb{Z}) = \{f, \phi\} \approx \mathbb{Z}_2$ as $\mathbb{Z}_2$ is cyclic so f is identity map which is identity of Aut $(\mathbb{Z})$, Aut $(\mathbb{Z})$ also cyclic.

Hence, correct option is (a).

(xiv). Any automorphism of the group ϕ under addition is of the form $x \mapsto qx$ for some $q \in \mathbb{Q}$. True or False?

[TIFR-2013]

Ans. True

Soln. Let f be isomorphism from $\mathbb{Q}$ to $\mathbb{Q}$. Let $1 \to q$ for same $q \in \mathbb{Q}$ then $m \to mq$ for $m \in \mathbb{Z}$

$$\mathbb{Q} = \left\{\frac{r}{s}; r \in \mathbb{Z}, s \in \mathbb{Z} - \{0\}, \gcd(r, s) = 1\right\}$$

Take $x \in \mathbb{Q} \Rightarrow x = \dfrac{r}{s}$ for some $r \in \mathbb{Z}$ and $s \in \mathbb{Z} - \{0\}$ then $r = sx$.

$f(r) = s\, f(x)$ by property of isomorphism as $r \in \mathbb{Z}$ so $f(r) = r\, f(1) = rq$.

$\Rightarrow \quad f(x) = \dfrac{r}{s} q = xq$; $f(x) = xq$ i.e. $x \to xq$

Hence all the isomorphism of the form $x \to qx$ for some $q \in \mathbb{Q}$.

(xv). The automorphism group Aut $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is abelian. True or False ?

[TIFR-2012]

Ans. False

Soln. Aut $(\mathbb{Z}_2 \times \mathbb{Z}_2) \approx GL_2(\mathbb{Z}_2)$

$GL_2(\mathbb{Z}_2)$ is non-abelian so Aut $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is non-abelian.

12. **First Isomorphism theorem :** A homomorphism $f : G \to G'$ is a mapping that preserve group operation. i.e., $f(a * b) = f(a) * f(b)\ \forall\, a, b$ in G

Kernal of homomorphism $f : G \to G'$ is defined as the set $\{x \in G \mid f(x) = e'$ where e' is identity of G$\}$ denoted $\ker \phi$.

Let f be a group homomorphism from G to G' then $\dfrac{G}{\ker \phi} \approx \phi(G)$.

SOLVED EXAMPLES

(i). Let G be additive group of integer I and G' be the multiplicative group of the fourth root of unity.

Let $f : G \to G'$ be a homomorphism mapping given by $f(n) = i^n$ where $i = \sqrt{-1}$ then the kernal of f is given by

(a) empty set (b) $\{4m : m \in I\}$

(c) $\{(2m)^2 + 1 : m \in I\}$ (d) $\{2m + 1 : m \in I\}$

[GATE-2000]

Soln. $\ker f = \{x \in G \mid f(x) = 1 \text{ in } G'\} = \{x \in G \mid f(x) = i^x = 1 \text{ in } G'\}$

$\Rightarrow \ker f = \{x \in G \mid x = 4m : m \in \mathbb{Z}\}$

Hence $\ker f = \{4m : m \in I\}$

Hence, correct option is (b).

(ii). Consider the group homomorphism $\phi : M_2(\mathbb{R}) \to (\mathbb{R})$ given by $\phi(A) = \text{trace}\,(A)$, then kernal of ϕ is isomorphic to which of the following group

(a) $\dfrac{M_2(\mathbb{R})}{\{A \in M_2(\mathbb{R}) \mid \phi(A) = 0\}}$ (b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$ (d) $GL_2(\mathbb{R})$ **[GATE-2014]**

Soln. By definition kernal of $\phi = \{A \in M_2(\mathbb{R}) \mid \phi(A) = 0\}$, then by first homomorphism theorem

$\dfrac{G}{\ker\phi} \approx \phi(G)$ as ϕ is onto so $\phi(G) = \mathbb{R}$

$\Rightarrow \dfrac{G}{\ker\phi} \approx \mathbb{R}$ and $M_2(\mathbb{R}) \approx \mathbb{R}^4$, So $\dfrac{\mathbb{R}^4}{\ker\phi} \approx \mathbb{R} \Rightarrow \ker\phi \approx \mathbb{R}^3$

Hence, correct option is (c).

(iii). Let G be group with the generators a and b given by $G = \langle a, b\,;\, a^4 = b^2 = 1, ba = a^{-1}b\rangle$. If $Z(G)$ denotes the center of G then $G \mid Z(G)$ is isomorphic to

(a) The trivial group (b) C_2, the cyclic group of order 2

(c) $C_2 \times C_2$ (d) C_4

[GATE-2006]

Soln. $G \approx D_4$. Dihedral group of order 8 then $|Z(G)| = 2$ so $O\left(\dfrac{G}{Z(G)}\right) = \dfrac{8}{2} = 4.$

$\dfrac{G}{Z(G)} \approx H, H$ is a group of order 4. So $\dfrac{G}{Z(G)} \approx C_2 \times C_2$ and C_4.

If $\dfrac{G}{Z(G)} \approx C_4$ then G is abelian but this is contradiction as G is non-abelian so $\dfrac{G}{Z(G)} \approx C_2 \times C_2.$

Hence, correct option is (c).

(iv). The number of group homomorphism from $\mathbb{Z}_3$ to $\mathbb{Z}_9$ are_____. **[GATE-2003]**

Ans. (3)

Soln. The number of group homomorphism from $\mathbb{Z}_m$ *to* $\mathbb{Z}_n$ is $\gcd(m, n)$. So $\gcd(3, 9) = 3$.

(v). There are *n*-homomorphism from the group $\frac{\mathbb{Z}}{n\mathbb{Z}}$ to the additive group of rational $\mathbb{Q}$. True or False ?

[TIFR-2011]

Ans. False

Soln. Let f be any homomorphism from $\frac{\mathbb{Z}}{n\mathbb{Z}}$ to $\mathbb{Q}$.

$f : \frac{\mathbb{Z}}{n\mathbb{Z}} \to \mathbb{Q}$ then $0 \to 0$. let $1 \to a$ then by property of homomorphism $O(a)/n$ but as $a \in \mathbb{Q}$ as in $\mathbb{Q}$ no element has finite order than 0 in addition. So only trivial homomorphism exist. Hence statement is false.

(vi). There exist a non-trivial group homomorphism from S_3 to $\frac{\mathbb{Z}}{3\mathbb{Z}}$. True or False ?

[TIFR-2011]

Ans. False

Soln. $S_3 \to \mathbb{Z}_3 = \{0, 1, 2\}$. Let $(1) \to 0$, $(12) \to a \;\Rightarrow\; o(a)|2$, there is no a in $\mathbb{Z}_3$ such that $o(a) = 2$. So only homomorphism is trivial homomorphism.

(vii). Let $G = R - \{0\}$ and $H = \{-1, 1\}$ be the group under multiplication. Then the map $\phi : G \to H$ defined by $\phi(x) = \frac{x}{|x|}$ is

(a) not a homomorphism
(b) a H homomorphism which is not onto
(c) an onto homomorphism which is not one-one
(d) an isomorphism

[GATE-2008]

Soln. Check homomorphism $\phi : G \to H$, consider $\phi(xy) = \frac{xy}{|xy|} = \frac{x}{|x|} \bullet \frac{y}{|y|}\ \forall\ x, y \in G = \phi(x) \bullet \phi(y)$

$\Rightarrow\ \phi$ is operation preserving.

For onto: ϕ is clearly onto every $x > 0$ will work for 1 and every $x < 0$ will work for –1.

For one-one: Let

$$\ker \phi = \{x \in G \mid \phi(x) = 1 \text{ in } H\}$$
$$= \{x \in G \mid \frac{x}{|x|} = 1 \text{ in } H\}$$
$$= \{x \in G \mid x > 0\} \neq \{e\} \text{ of } G$$

Hence ϕ is not one-one as $\ker \phi$ is not trivial.

Hence, correct option is (c).

(viii). Consider the following statement P and Q

P : If H is a normal subgroup of order 4 of symmetric group S_4 then S_4/H is abelian

Q : If $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is quaternion group then $Q\big|_{\{-1,1\}}$ is abelian

Which of the following is true ?

(a) both P and Q (b) only P (c) only Q (d) neither P nor Q

[GATE-2016]

Soln. (i) $\frac{Q_8}{\{1,-1\}} \cong G'$ then $|G'| = \frac{8}{2} = 4$ and every group of order 4 is abelian $\Rightarrow Q$ is true.

(ii) $\frac{S_4}{H} \cong G' = \frac{24}{4} = 6$ then G' is non abelian .

Hence, correct option is (c).

(ix). Let G be a group whose presentation is $G = \{x, y \mid x^5 = y^2 = e,\ x^2 y = y x\}$, then G is isomorphic to

(a) $\mathbb{Z}_5$ (b) $\mathbb{Z}_{10}$ (c) $\mathbb{Z}_2$ (d) $\mathbb{Z}_{30}$ **[GATE-2016]**

Soln. Given that, $G = \{x, y \mid x^5 = y^2 = e,\ x^2 y = y x\}$, in the group G only satisfy the given condition by $\mathbb{Z}_2$

Hence, correct option is (c).

(x). The number of group homomorphisms from the symmetric group S_3 to the additive group $\mathbb{Z}/6\mathbb{Z}$ is

(a) 1 (b) 2 (c) 3 (d) 0

[CSIR-NET-2013-(II)]

Soln. The normal subgroup of S_3 are $\{e\}$ and A_3.

We know that the kernal of homomorphism is normal subgroup.

$\therefore$ Number of group homomorphism from S_3 to $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is 2.

Hence, correct option is (b).

(xi). The number of non-trivial ring homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{28}$ is

(a) 1 (b) 3 (c) 4 (d) 7

[CSIR-NET-2012-(I)]

Soln. Given that, $f : \mathbb{Z}_{12} \to \mathbb{Z}_{28}$ and idempotent element in $\mathbb{Z}_{28}$ is $\{0, 1, 8, 21\}$, then make the order same both side. i.e., $f : \underset{12\times 7}{\mathbb{Z}_{12}} \to \underset{28\times 3}{\mathbb{Z}_{28}}$, $\{0, 1, 8, 21\}$

$7 \mid 0,\quad 7 \nmid 1,\quad 7 \nmid 8,\quad 7 \mid 21$

Therefore, number of ring homomorphism $f : \mathbb{Z}_{12} \to \mathbb{Z}_{28}$ is 2, but number of non-trivial ring homomorphism is 1.

Hence, correct option is (a).

(xii). Let G be a group of order 77. Then the center of G is isomorphic to

(a) $\mathbb{Z}_1$ (b) $\mathbb{Z}_7$ (c) $\mathbb{Z}_{11}$ (d) $\mathbb{Z}_{77}$

[CSIR-NET-2011-(I)]

Soln. $o(G) = 77 = 7 \cdot 11$ but $7 \nmid (11-1) \Rightarrow G$ is abelian

$\Rightarrow Z(G) = G$

$\Rightarrow Z_{77}$

Hence, correct option is (d).

(xiii). Suppose G is group of order 29, then which of the following is true ?

(a) G is not abelian

(b) G has no subgroup other than $\{e\}$ and G
(c) There is a group H of order 29 which is not isomorphic of G
(d) G is a subgroup of a group of order 30 **[CSIR-NET/JRF : 2002]**

Soln. $O(G) = 29$, which is prime

We know that every group of prime order are cyclic and every positive divisor of G is subgroup of G. So positive divisor of G is 1, 29

$\Rightarrow$ G has only two subgroup one is identity and another is G itself.

Hence, correct option is (b).

(xiv). Consider the group $G = \frac{\mathbb{Q}}{\mathbb{Z}}$ where $\mathbb{Q}$ and $\mathbb{Z}$ are the groups of rational numbers and integers respectively. Let n be a positive integer. Then is there a cyclic subgroup of order n?

(a) not necessarily (b) yes, a unique one
(c) yes, but not necessarily a unique one (d) never

[CSIR-NET/JRF : June-2012]

Soln. Every proper subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic and unique.

Hence, correct option is (b).

Some Important Theorems

1. Every infinite cyclic group $G = \langle a \rangle$ isomorphic to $(\mathbb{Z}, +)$.
2. Every finite cyclic group $G = \langle a \rangle$ of order n is isomorphic to $(\mathbb{Z}_n, +_n)$.
3. Let G and G' are two cyclic group of same order then they are isomorphic and isomorphism is which map generator of G onto generator G'.
4. A group G is abelian iff $f(x) = x^{-1}$ is an automorphism.
5. Every odd order abelian group has non-trivial automorphism, namely $f(x) = x^{-1}\ \forall\ x \in G$.

Important properties of Homomorphism/Isomorphism

6. Let G be a group Aut (G) set of all automorphism is subgroup of S_G group of permutation on G under composition of function.
7. Inn (G), set of all inner automorphism is normal subgroup of Aut (G).
8. Corresponding elements of centre $Z(G)$ of G there is only one inner automorphism which is identity map.
9. Index of centre $Z(G) = O$ (Inn (G)
10. Let G be a finite group and $Z(G)$ be centre of G then $\frac{G}{Z(G)} \cong Inn\ (G)$.
11. If G is an abelian then identity function is only the inner automorphism G onto itself.
12. G is abelian iff the function $f : G \to G$ such that $f(x) = x^2\ \ \forall\ x \in G$ is a homomorphism.
13. G abelian group $f(x) = x^n,\ \forall\ x \in G$. Homomorphism for each $n \in \mathbb{N}$.
14. If $G/Z(G)$ is cyclic then G is abelian.

PRACTICE SET - 5

Multiple Correct Answer Type Questions :

1. Choose the incorrect statement(s)

(a) The class of all automorphisms of a group contains the class of all isomorphism of the group.
(b) The class of all isomorphism of a group contains the class of all homomorphism of the group.
(c) The class of all homomorphism of a group contains the class of all onto homomorphisms of the group.
(d) The class of all onto homomorphism on a group contains the class of all onto functions of the group.

2. For the following pair of groups which one is homomorphism

(a) $f(x) = -x$, where $f:(\mathbb{R},+) \longrightarrow (\mathbb{R},+)$

(b) $f(x) = x^2$, where $f:(\mathbb{R}-\{0\},\bullet) \longrightarrow (\mathbb{R},+)$

(c) $f(x) = x+1$, where $f:(\mathbb{Z},+) \longrightarrow (\mathbb{Z},+)$

(d) $f(x) = \dfrac{x}{q}$ ($q \neq 0$ is whole number), where $f(\mathbb{Z},+) \longrightarrow (\mathbb{Q},+)$

3. Select the correct statements :

(a) If $\text{Aut}(G_1) \cong \text{Aut}(G_2)$ and G_1 is infinite group then G_2 is also infinite.

(b) If $\text{Aut}(G_1) \cong \text{Aut}(G_2)$ and G_1 is finite then G_2 is also finite.

(c) $G_1 \not\cong G_2$ then $\text{Aut}(G_1) \not\cong \text{Aut}(G_2)$

(d) If $\text{Aut}(G_1)$ non abelian then G_1 non cyclic.

4. Let $G = \mathbb{R}-\{0\}$ and $H = \{-1,1\}$ be groups under multiplication, then the map $\phi: G \to H$ defined by $\phi(x) = \dfrac{x}{|x|}$ is

(a) Not a homomorphism.
(b) A one-one homomorphism which is not onto.
(c) An onto homomorphism, which is not one-one.
(d) An isomorphism.

5. Which statement is/are correct

(a) Two cyclic groups of same order are isomorphic.

(b) Isomorphic image of a cyclic group is isomorphic.

(c) Two non-abelian groups of order 8 are isomorphic.

(d) None of these.

6. Let $A = \{f: Q_8 \to S_3 : f \text{ is a homomorphism}\}$ and

$B = \{f: Q_8 \to S_3 : |f \text{ is a one one homomorphism}\}$, $C = \{f: Q_8 \to S_3 : |f \text{ is a onto homomorphism}\}$

then

(a) $|A| = 10$ (b) B is empty (c) C is empty (d) $|A| = 6$

Single Correct Answer Type Questions :

7. Let G be a cyclic group of order n with β as a generator. Let m be a positive integer greater than one, less than n and relatively prime to n. Let f be a mapping defined on G by setting $f(\beta) = \beta^m \ \forall \ \beta \in G$, then choose correct statement

(a) f is a homomorphism, but not an isomorphism

(b) f is a not homomorphism

(c) f is inner automorphism

(d) f is an automorphism, but not an inner automorphism

8. Let $H = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $K = \mathbb{Z}_3 \times \mathbb{Z}_4$ then

(a) H is isomorphic to K since both are cyclic

(b) H is not isomorphic to K since, K is cyclic where H is not.

(c) H is not isomorphic to K since there is no homomorphism from H to K.

(d) None of these

9. Let $\mathbb{Z}$ be the group of integers under addition then $\text{Aut}(\mathbb{Z})$ is isomorphic to

(a) $\mathbb{Z}_2$ (b) $\mathbb{Z}$ (c) $\mathbb{Z}\times\mathbb{Z}$ (d) $\mathbb{Z}_2\times\mathbb{Z}_2$

10. Let ϕ be a homomorphism from a finite group G into the group C^* of non-zero complex numbers. Then for all $g\in G$.

(a) $\phi(g)=1$ (b) $|\phi(g)|=1$

(c) $\phi(g)$ is purely imaginary (d) $\phi(g)$ is real

11. Number of homomorphism from $\mathbb{Z}_{6k}$ $(k\in\mathbb{N}$ fixed$)$ into G is non abelian group of order 6.

(a) 6 (b) $k\cdot\phi(6)$ (c) $6\cdot\phi(k)$ (d) $6\cdot\phi(6)$

12. $f:(\mathbb{Q},+)\longrightarrow(\mathbb{Q},+)$ where $(\mathbb{Q},+)$ is additive abelian group of rational number if f is non trivial homomorphism, then

(a) f is $1-1$ (b) f is onto (c) f is bijective (d) Neither $1-1$ nor onto

Numerical Answer Type Questions:

13. Number of one-one homomorphism from $\mathbb{Z}_n\to\mathbb{Z}$, where $(n>2)$ = ??

14. Number of homomorphism from $\mathbb{Z}$ onto S_n where $(n>2)$ = ??

15. Number of subgroups which is isomorphic to $\mathbb{Z}$ = ??

16. $|\text{Aut}\,(\mathbb{Z}_2\times\mathbb{Z}_2)|$ = ??

17. Number of homomorphism from $\mathbb{Z}_9$ into $\mathbb{Z}_6$ are = ??

18. Number of groups upto isomorphic of order 122.

19. Number of groups upto isomorphic of order 15.

20. Order of inner automorphism for S_3 = ??

SOLUTIONS PRACTICE SET - 5

Multiple Correct Answer Type Questions :

1. (a), (b), (c)

2. (a) and (d)
Since (b) discarded by $f(x+y)\neq f(x)+f(y)$ and (c) discarded because identity does not go to identity.

3. (d)
Since $\mathbb{Z}\not\cong\mathbb{Z}_4$
But $\text{Aut}(\mathbb{Z})\cong\text{Aut}(\mathbb{Z}_4)\cong\mathbb{Z}_2$

4. (c) as $\begin{pmatrix}1\to1\\-1\to1\end{pmatrix}$

5. (a), (b)
Since Q_8 and D_4 both are of same order but not isomorphic to each other.

6. (a), (b) and (c)

Single Correct Answer Type Questions :

7. (d)

8. (b), $H = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $K = \mathbb{Z}_3 \times \mathbb{Z}_4$ are not isomorphic since H is not cyclic but K is cyclic

9. (a), $\mathbb{Z}_2$

10. (b) $|\phi(g)| = 1$, since every elements of a finite group has order finite and on which they mapped by ϕ also has order finite. And in $\mathbb{C}^*$ each elements of order finite iff $|x| = 1$.

11. (a)

12. (a), (b), (c)

Numerical Answer Type Questions

13. '0' since in $\mathbb{Z}$ there is no elements of finite order other than zero element.

14. Zero, since $\mathbb{Z}$ is cyclic and S_n is non-cyclic.

15. Infinite; infact every subgroup except '0' subgroup always isomorphic to $\mathbb{Z}$.

16. 6 since Aut $(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$

17. 3

18. 2 $O(G) = 122 = 2 \times 61 = p \times q$ form, total number of non isomorphic groups of order 122 = 2 i.e., $\mathbb{Z}_{pq}$ and D_q

19. 1 since $3 \nmid (5-1)$ (break p.q. form)

20. 6 since $\text{inn}(S_3) \cong S_3$