Chapter 5

Orthogonal Trajectories

ORTHOGONAL TRAJECTORY OF A GIVEN CURVE (CARTESIAN FORM)

A curve of family or curves $\phi(x, y, c) = 0$ which cuts every member of a given family of curves $f(x, y, c) = 0$ according to a fixed rule is called a trajectory of the family of curves.

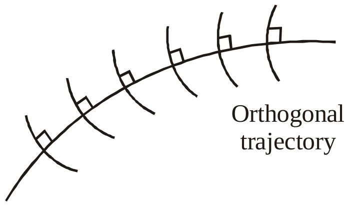


Figure-1

If we consider only the trajectories cutting each member of family of curves $f(x, y, c) = 0$ at a constant angle, then the curve which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family.

In order to find out the orthogonal trajectories the following steps are taken.

Step-1: Let $f(x, y, c) = 0$ be the equation where c is an arbitrary parameter.

Step-2: Differentiate the given equation w.r.t. x and eliminate c.

Step-3: Substitute $\frac{-dx}{dy}$ for $\frac{dy}{dx}$ in the equation obtained in step 2.

Step-4: Solve the differential equation obtained from step 3.

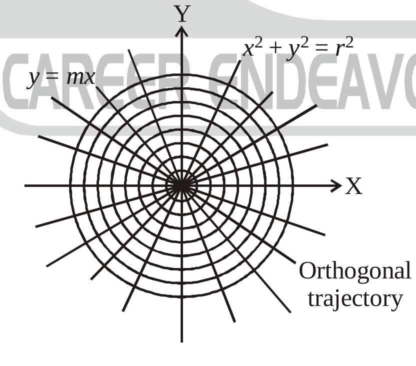


Figure-2

Example-1

Find the orthogonal trajectory of family of straight lines passing through the origin.

Soln. Family of straight lines passing through the origin is $y = mx$. ...(i)

where 'm' is an arbitrary constant.

Differentiating w.r.t. x, we get $\frac{dy}{dx} = m$ ...(ii)

Eliminating 'm' from (i) and (ii), we get $y = \frac{dy}{dx}x$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get $y = -\frac{dx}{dy}x \quad x\,dx + y\,dy = 0$

Integrating both sides, we get $\frac{x^2}{2} + \frac{y^2}{2} = c$

$\Rightarrow \; x^2 + y^2 = 2c$

Which is the required orthogonal trajectory.

Example-2

Find the orthogonal trajectories of the curve $y = ax^2$.

Soln. Let $y = ax^2$ ...(i)

Differentiating equation (i), we get $\frac{dy}{dx} = 2ax$ ...(ii)

Eliminating a from equations (i) and (ii), we get $\frac{dy}{dx} = \frac{2y}{x^2}x$ ...(iii)

Putting $-\frac{dx}{dy}$ in equation (iii) in the place of $\frac{dy}{dx}$, we get $-\frac{dx}{dy} = \frac{2y}{x}$

$\Rightarrow \; -x dx = 2y\,dy \qquad \Rightarrow \; -\frac{x^2}{2} = \frac{2y^2}{2}$

$\Rightarrow \; 2y^2 + x^2 = 0$

This is the family of required orthogonal trajectories.

Example-3

Find the orthogonal trajectories of the hyperbola $xy = c$.

Soln. The equation of the given family of curves is $xy = c$ ...(i)

Differentiating (i) w.r.t. x, we get $x\frac{dy}{dx} + y = 0$ ...(ii)

Substituting $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in (ii), we get $-x\frac{dx}{dy} + y = 0$ ...(iii)

This is the differential equation of the orthogonal trajectories of given family of hyperbolas. (iii) can be rewritten as xdx = ydy, which on integration given $x^2 - y^2 = c$.

This is the required family of orthogonal trajectories.

Example-4

Find the orthogonal trajectories of the curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, λ being the parameter of the family.

Soln. Let $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ ...(i)

Differentiating (i) w.r.t. x, we have $\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$ ...(ii)

From (i) and (ii) we have to eliminate λ.

Now (ii) gives, $\lambda = -\left(\dfrac{b^2x + a^2y\dfrac{dy}{dx}}{x + y\dfrac{dy}{dx}}\right)$

$$\Rightarrow\ a^2 + \lambda = \frac{(a^2-b^2)x}{x+y\dfrac{dy}{dx}};\ b^2 + \lambda = -\frac{(a^2-b^2)y\dfrac{dy}{dx}}{x+y\dfrac{dy}{dx}}$$

Substituting these values in (i), we get $\left(x - y\dfrac{dx}{dy}\right)\left(x + y\dfrac{dy}{dx}\right) = a^2 - b^2$ ...(iii)

is the differential equation of the given family of curves.

Replacing $\dfrac{dy}{dx}$ to $-\dfrac{dx}{dy}$ in (iii), we obtain $\left(x + y\dfrac{dy}{dx}\right)\left(x - y\dfrac{dx}{dy}\right) = a^2 - b^2$ ...(iv)

which is the same as (iii). Thus we see that the family (i) is self orthogonal. i.e., every member of the family (i) cuts every other member of the same family orthogonally.

Example-5

Find the orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$ where 'a' is a parameter.

Soln. Here, $x^2 + y^2 - ay = 0$. Differentiating w.r.t. x, we get $2x + 2y\dfrac{dy}{dx} - a\dfrac{dy}{dx} = 0$

$$\Rightarrow\ 2x + 2y\frac{dy}{dx} - \frac{x^2+y^2}{y}\frac{dy}{dx} = 0 \qquad \left(\because\ x^2 + y^2 - ay = 0 \Rightarrow a = \frac{x^2+y^2}{y}\right)$$

$\Rightarrow\ 2x + \dfrac{y^2-x^2}{y}\dfrac{dy}{dx} = 0$. This is the differential equation of the given circles.

$\therefore$ The equation of orthogonal trajectories is $2x + \dfrac{y^2-x^2}{y}\cdot\left(-\dfrac{dx}{dy}\right) = 0$

$\left(\text{putting } -\dfrac{dx}{dy} \text{ in place of } \dfrac{dy}{dx}\right)$

$\Rightarrow\ 2xydy + (x^2 - y^2)dx = 0$

It is a homogeneous equation.

Put $y = \upsilon x$, then $\dfrac{dy}{dx} = \upsilon + x\dfrac{d\upsilon}{dx} \Rightarrow 2x.\upsilon x.\left(\upsilon + x\dfrac{d\upsilon}{dx}\right) + x^2 - \upsilon^2x^2 = 0$

or $2\upsilon\left(\upsilon + x\dfrac{d\upsilon}{dx}\right) + 1 - \upsilon^2 = 0 \Rightarrow 1 + \upsilon^2 + 2\upsilon x\dfrac{d\upsilon}{dx} = 0 \Rightarrow \dfrac{dx}{x} + \dfrac{2\upsilon}{1+\upsilon^2}d\upsilon = 0;$

Integrating, we get $\ell n\ x + \ell n\ (1+\upsilon^2) = \ell n\ c$

$\Rightarrow\ x(1+\upsilon^2) = c \Rightarrow x\left(1 + \dfrac{y^2}{x^2}\right) = c,$ i.e., $x^2 + y^2 = cx$

Which is the required family of orthogonal trajectories.

Previous Year Solved Problems

Example-6

The trajectories of the system of differential equations $dx/dt = y$ and $dy/dt = -x$ are **[D.U. 2014]**

(a) ellipses (b) hyperbolas (c) circles (d) spirals

Soln. $\dfrac{dx}{dt} = y$ and $\dfrac{dy}{dt} = -x$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x}{y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow ydy + xdx = 0 \Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$$

$\Rightarrow x^2 + y^2 = c^2$ which is equation of circle

Option (c) is Correct

Example-7

Let γ be the curve which passes through (0, 1) and intersects each curve of the family $y = c\,x^2$ othogonally. Then γ also passes through the point

(a) $\left(\sqrt{2}, 0\right)$ (b) $\left(0, \sqrt{2}\right)$ (c) (1, 1) (d) (–1, 1) **[GATE-2016]**

Soln. $y = cx^2 \Rightarrow c = \dfrac{y}{x^2} \Rightarrow \dfrac{dy}{dx} = c.2x$

We need to remove arbitary constnat c

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x^2}2x = \frac{2y}{x} \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$$

For othogonal trajectories, we replace

$$\frac{dy}{dx} by \frac{-dx}{dy} \Rightarrow \frac{-dx}{dy} = \frac{2y}{x} \Rightarrow 2ydy + xdx = 0$$

$\Rightarrow \dfrac{2y^2}{2} + \dfrac{x^2}{2} = \dfrac{c^2}{2} \Rightarrow 2y^2 + x^2 = c^2$ which is required equation of y.

Since, γ passes through (0, 1)

$2\times1^2 + 0^2 = c^2 \Rightarrow c^2 = 2$

So, $\boxed{2y^2 + x^2 = 2}$

Now, $x = \sqrt{2},\ y = 0$ satisfies above equation of y.

Option (a) is Correct

$x = 0, y = \sqrt{2}$ doesnot satisfy the equation of y.

$x = 1, y = 1$ doesnot satisfy the equation of y.

$x = -1, y = 1$ doesnot satisfy the equation of y.

Option (b), (c), (d) are not Correct.

Example-8

The orthogonal trajectories of the one parameter family of curves $y^2 = 4k(k+x)$ where k is an arbitrary constant is **[ISM-2015]**

(a) $x^2 = 2c(c+y)$ (b) $y = 4c(c+x)$ (c) $x^2 = 4c(c+y)$ (d) $y^2 = 4c(c+x)$

Soln. $y^2 = 4k(k+x)$ ...(i)

$$\Rightarrow 2y\frac{dy}{dx} = 4k \Rightarrow \frac{y}{2}\frac{dy}{dx} = k$$

To remove arbitary Constant k

Put k in equation (i)

$$\Rightarrow y^2 = 4\frac{y}{2}\left(\frac{dy}{dx}\right)\left(\frac{y}{2}\frac{dy}{dx}+x\right) \Rightarrow y^2 = 2y\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx}+x\right)$$

$$\Rightarrow y = 2\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx}+x\right) \Rightarrow y = y\left(\frac{dy}{dx}\right)^2 + 2xc\left(\frac{dy}{dx}\right) \qquad \text{(iii)}$$

To get orthogonal trajectary, replace

$\frac{dy}{dx} by \frac{-dx}{dy}$

$$\Rightarrow y = y\left(\frac{-dx}{dy}\right)^2 - 2x\left(\frac{dx}{dy}\right) \Rightarrow y = y\left(\frac{dx}{dy}\right)^2 - 2x\left(\frac{dx}{dy}\right)0$$

$$\Rightarrow y + 2x\frac{dx}{dy} = y\left(\frac{dx}{dy}\right)^2 \Rightarrow y\frac{dy}{dx} + 2x = y\left(\frac{dx}{dy}\right) \Rightarrow \boxed{y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y} \qquad \text{...(iv)}$$

equation (iv) is same as equation (iii)

Given equation is self orthogonal samily of carves.

The orthogonal tracjector will be

$\boxed{y^2 = 4c(c+x)}$ which c is orthogonal constant

Option (d) is Correct

Example-9

6. The orthogonal trajectory of the hyperbola $x^2 - y^2 = c$ is **[SAU-2015]**

(a) $xy = c$ (b) $xy^2 = c$ (c) $x^2y = c$ (d) None of the above

Soln. $x^2 - y^2 = c^2$

$$2x - 2y\frac{dy}{dx} = 0$$

To get orthogonal trajectory, we replace $\frac{dy}{dx}$ by $-\frac{dy}{dx}$

$$\Rightarrow\ 2x-2y\left(\frac{-dx}{dy}\right)=0\ \Rightarrow\ 2x+2y\frac{dx}{dy}=0\ \Rightarrow\ 2y\frac{dx}{dy}=-2x$$

$$\Rightarrow\frac{dx}{x}=-\frac{dy}{y}\ \Rightarrow\ \int\frac{dx}{x}+\int\frac{dy}{y}=0\ \Rightarrow\ \log|x|+\log|y|=\log(c)\Rightarrow\ \log|xy|=\log|c|\ \Rightarrow\ \boxed{xy=c}$$

Option (a) is Correct

Example-10

Let c be an arbitrary non-zero constant. Then the orthogonal family of curves to the family $y(1-cx)=1+cx$ is **[H.C.U.-2015]**

(a) $3y-y^3+3x^2=\text{constant}$

(b) $3y+y^3-3x^2=\text{constant}$

(c) $3y-y^3-3x^2=\text{constant}$

(d) $3y+y^3+3x^2=\text{constant}$

Soln. $y(1-cx)=1+cx\ \Rightarrow\ y-ycx=1+cx\ \Rightarrow\ y-1=ycx+cx$

$$\Rightarrow\ y-1=c(xy+x)\ \Rightarrow\ \frac{y-1}{xy+x}=c\Rightarrow\ \frac{y-1}{(y+1)x}=c$$

Differentiating both side w.r.t x

$$\Rightarrow\frac{d}{dx}\left(\frac{y-1}{(y+1)x}\right)=0\ \Rightarrow\ \frac{x(y+1)\frac{dy}{dx}-(y-1)\left[x\frac{dy}{dx}+(y+1)\right]}{x^2(y+1)^2}=0$$

$$\Rightarrow\ x(y+1)\frac{dy}{dx}-(y-1)\left[x\frac{dy}{dx}+y+1\right]=0\ \Rightarrow\ x(y+1)\frac{dy}{dx}-x(y-1)\frac{dy}{dx}-\left(y^2-1\right)=0$$

$$\Rightarrow\ x\frac{dy}{dx}(y+1-y+1)-\left(y^2-1\right)=0\ \Rightarrow\ 2x\frac{dy}{dx}-y^2+1=0$$

Toget orthogonal trajectories, we replace

$$\frac{dy}{dx}\,by\,\frac{-dx}{dy}$$

$$2x\left(\frac{-dx}{dy}\right)-y^2+1=0\ \Rightarrow\ \frac{-2xdx}{dy}=y^2-1$$

$$\Rightarrow\ -2xdx=\left(y^2-1\right)dy\ \Rightarrow\ 2\int xdx=\int\left(y^2-1\right)dy$$

$$\Rightarrow\ -2\frac{x^2}{2}=\frac{y^3}{3}-y+c_1\ \Rightarrow\ -x^2=\frac{y^3}{3}-y+c_1$$

$$\Rightarrow\ 3x^2=y^3-3y+3c_1\ \Rightarrow\ \boxed{3y-y^3-3x^2=\text{constant}}$$

Option (c) is Correct

Example-11

The orthogonal trajectory to the family of circle $x^2 + y^2 = 2cx$ (c is orbitary) is desoribed by the differential equation. **[CUCET-2017]**

(a) $\left(x^2 + y^2\right) y' = 2xy$

(b) $\left(x^2 - y^2\right) y' = 2xy$

(c) $\left(y^2 - x^2\right) y' = xy$

(d) $\left(y^2 - x^2\right) y' = 2xy$

Soln. $x^2 + y^2 = 2cx$

dividing both side by x $\Rightarrow x + \dfrac{y^2}{x} = 2c$

Differentiating both side w.r.t x

$$\Rightarrow 1 + \frac{2xy\dfrac{dy}{dx} - y^2}{x^2} = 0 \quad \Rightarrow \frac{x^2 + 2xy\dfrac{dy}{dx} - y^2}{x^2} = 0 \Rightarrow x^2 + 2xy\frac{dy}{dx} - y^2 = 0$$

To get, orthogonal trajectaries, we replace $\dfrac{dy}{dx} by \dfrac{-dx}{dy}$

$$\Rightarrow x^2 + 2xy\left(\frac{-dx}{dy}\right) - y^2 = 0 \Rightarrow x^2 - y^2 - 2xy\frac{dx}{dy} = 0 \Rightarrow x^2 - y^2 = 2xy\frac{dx}{dy} \Rightarrow \left(x^2 - y^2\right) y' = 2xy$$

Option (b) is Correct

Example-12

The orthogonal trajectories of the family of rectangular hyperbolas $y = c_1 / x$ is **[HCU-2018]**

(a) $y^2 - x^2 = c$

(b) $y^2 + x^2 = c$

(c) $x^2 y^2 = c$

(d) $\dfrac{x^2}{y^2} = c$

Soln. $y = c_1 / x \Rightarrow xy = c_1$

differenting w.r.t $x \Rightarrow y + x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

To get orthogonal trajectories replace $\dfrac{dy}{dx} by \dfrac{-dx}{dy}$

$$\Rightarrow \frac{-dx}{dy} = \frac{-y}{x} \Rightarrow xdx = ydy \Rightarrow x^2 + c = y^2 \Rightarrow y^2 - x^2 = c$$

Option (a) is Correct

Example-13

The orthogonal trajectories of the curves xy = c is (*a* being a constant) **[BHU-2018]**

(a) $x^2+y^2=a^2$ (b) $x^2-y^2=a^2$

(c) $x^2+2y^2=a^2$ (d) $2x^2+y^2=a^2$

Soln. $y=c_1/x \Rightarrow xy=c_1$

differenting w.r.t $x \Rightarrow y+x\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{y}{x}$

To get orthogonal trajectories replace $\dfrac{dy}{dx} by \dfrac{-dx}{dy}$

$$\Rightarrow \frac{-dx}{dy}=\frac{-y}{x} \Rightarrow xdx=ydy \Rightarrow x^2+c=y^2 \Rightarrow y^2-x^2=c$$

Option (a) is Correct

Example-14

The orthogonal trajectories of the family of curves $x^2-y^2=a^2$ is **[CUCET-2018]**

(a) $x^2+y^2=c^2$ (b) $\dfrac{x}{y}=c$ (c) $xy=c$ (d) none of these

Soln. $x^2-y^2=a^2$

$2x-2y\dfrac{dy}{dx}=0$

For orthogonal trajectry

$$\Rightarrow 2x-2y\left(\frac{-dx}{dy}\right)=0 \Rightarrow 2x+2y\frac{dx}{dy}=0 \Rightarrow \frac{dx}{dy}=\frac{-x}{y}$$

$$\Rightarrow \frac{dx}{x}+\frac{dy}{y}=0 \Rightarrow \log|x|+\log|y|=\log c \Rightarrow \log|xy|=\log c \Rightarrow xy=c$$

Option (c) is Correct

Example-15

If the orthogonal trajectories of the family of ellipses $x^2+2y^2=c_1, c_1>0$, are given by $y=c_2x^\alpha, c_2\in\mathbb{R}$, then $\alpha=$____________ **[GATE-2017]**

Soln. $x^2+2y^2=c_1 \Rightarrow 2x+4y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{2y}$

For orthogonal trajectary we solve

$$\frac{-dx}{dy}=-\frac{-x}{2y} \Rightarrow 2\frac{dx}{x}=\frac{dy}{y} \Rightarrow 2\int\frac{dx}{x}=\int\frac{dy}{y} \Rightarrow 2\log|x|=\log|y|-\log|c_2| \Rightarrow \boxed{y=c_2x^2}$$

So, $\boxed{\alpha=2}$.

Concept Application Exercise

1. Find the orthogonal trajectories of the family of co-axial circles $x^2 + y^2 + 2gx + c = 0$ where g is the parameter.

2. Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$; where a is the variable parameter.

3. Find the orthogonal trajectories of the family of parabolas $y = ax^2$.

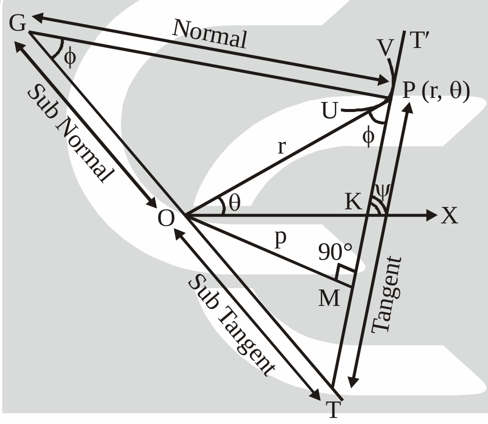
4. Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$; where λ is an arbitrary parameter.

5. Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

6. Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.
   (i) $\cos y = a e^{-x}$
   (ii) $x^k + y^k = a^k$

7. Find the trajectories orthogonal to ellipse having a common major axis equal to $2a$.

ORTHOGONAL TRAJECTORIES IN POLAR FORM F(r, θ, c) = 0



(i) Form its differential equation in the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.

(ii) Replace in this differential equation, $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$.

[$\because$ for the given curve through $P(r, \theta)$ $\tan\phi = r d\theta/dr$ and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$r\frac{d\theta}{dr}$ is to be replaced by $-\frac{1}{r}\frac{dr}{d\theta}$ or $\frac{dr}{d\theta}$ is to be replaced by $-r^2\frac{d\theta}{dr}$].

(iii) Solve the differential equation of the orthogonal trajectories

i.e. $f(r, \theta, -r^2 d\theta/dr) = 0$.

Example-16

Find the orthogonal trajectory of the cardioids $r = a(1-\cos\theta)$.

Soln. Differentiating $r = a(1-\cos\theta)$ ...(i)

with respect to θ, we get $\dfrac{dr}{d\theta} = a\sin\theta$ ...(ii)

Eliminating a from (i) and (ii), we obtain

$$\frac{dr}{d\theta}\cdot\frac{1}{r} = \frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2}$$ which is the differential equation of the given family.

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \text{ or } \frac{dr}{r} + \tan\frac{\theta}{2}d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)\,d\theta}{\cos\theta/2}$$

Integrating, $\log r = 2\log\cos\theta/2 + \log c$

$$r = c\cos^2\theta/2 = \frac{1}{2}c(1+\cos\theta) \text{ or } r = a'(1+\cos\theta)$$

which is the required orthogonal trajectory.

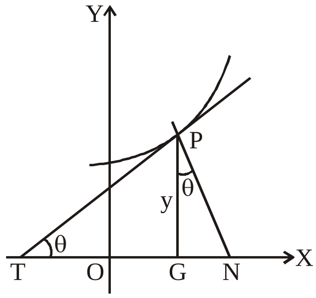
FINDING EQUATION OF A CURVE WHOSE GEOMETRICAL PROPERTIES ARE GIVEN

The following properties of a curve are sometimes very useful in determining the equation of a curve. Using these properties, first the differential equation of the curve is formed and then this differential equation is solved to get the equation of the curve.

Let the tangent and normal at a point $P(x, y)$ on the curve $y = f(x)$, meet the X-axis at T and N respectively. If G is the foot the ordinate at P, then TG and GN are called the Cartesian subtangent and subnormal, while the lengths PT and PN are called the lengths of the tangent and normal respectively.

If PT make an angle θ with X-axis, then $\tan\theta = dy/dx$. From the figure we can find that :

- Subtangent $= TG = y\cot\theta = \dfrac{y}{\left(\dfrac{dy}{dx}\right)}$
- Subnormal $= GN = y\tan\theta = y\dfrac{dy}{dx}$
- Length of the tangent $= PT = y\operatorname{cosec}\theta = y\sqrt{1+\cot^2\theta} = \dfrac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}$
- Length of the normal $= PN = y\sec\theta = y\sqrt{1+\tan^2\theta} = y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$



- Equation of tangent at $P(x, y) \equiv Y - y = \frac{dy}{dx}(X - x)$
- Equation of normal at $P(x, y) \equiv Y - y = -\frac{dx}{dy}(X - x)$
- Length of radius vector $OP = \sqrt{x^2 + y^2}$

The following examples will illustrate the concept of forming and solving the differential equations of the curves whose geometrical properties are given.

Example-17

The slope of curve passing through (4, 3) at any point is the reciprocal of twice the ordinate at that point. Show that the curve is a parabola.

Soln. The slope of the curve is the reciprocal of twice the ordinate at each point of the curve. Using this property, we can define the differential equation of the curve i.e.

$$\text{slope} = \frac{dy}{dx} = \frac{1}{2y}$$

Integrate both sides to get :

$$\int 2y\, dy = \int dx \quad \Rightarrow \quad y^2 = x + C$$

As the required curve passes through (4, 3), it lies on it.

$$\Rightarrow \; 9 = 4 + C \quad \Rightarrow \quad C = 5$$

So the required curve is : $y^2 = x + 5$ which is parabola.

Example-18

Find the equation of the curve passing through (2, 1) which has constant subtangent.

Soln. The length of subtangent is constant. Using this property, we can define the differential equation of the curve i.e.

$$\text{subtangent} = \frac{y}{y'} = k \text{ where } k \text{ is a constant}$$

$$\Rightarrow \; k\frac{dy}{dx} = y$$

Integrate both sides to get :

$$\int \frac{kdy}{y} = \int dx \quad \Rightarrow \quad k \log y = x + C$$

where C is an arbitrary constant.

As the required curve through (2, 1), it is lies on it.

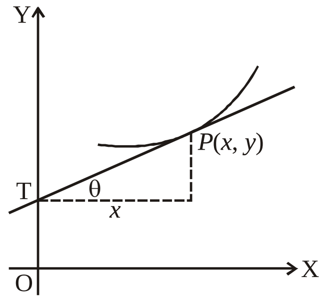
$\Rightarrow \; 0 = 2 + k \quad \Rightarrow \quad C = -2 \Rightarrow$ the equation of the curve is : $k \log y = x - 2$.

Note that above equation can also be put in the form $y = Ae^{Bx}$.

Example-19

Find the curve through (2, 0) so that the segment of tangent between point of tangency and y-axis has a constant length equation to 2.

Soln. The segment of the tangent between the point of tangency and y-axis has a constant length = PT = 2.



Using this property, we can define the differential equation of the curve i.e.

$$PT = x\sec\theta = x\sqrt{1+\tan^2\theta} = x\sqrt{1+y'^2} \Rightarrow y = \pm 2\left(\log\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2}\right) + C_1$$

As (2, 0) lies on the curve, it should satisfy its equation i.e. $C_1 = 0$

$$\Rightarrow x\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2 \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{4}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\sqrt{\frac{4-x^2}{x^2}}$$

Integrate both sides to get :

$$\Rightarrow y = \pm\int\sqrt{\frac{4-x^2}{x^2}}dx + C_1$$

Put $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$$\Rightarrow y = \pm 2\int\frac{\cos^2\theta}{\sin\theta}d\theta + C_1$$

$$= \pm 2\int(\operatorname{cosec}\theta - \sin\theta)\,d\theta + C_1$$

$$= \pm(2\log|\operatorname{cosec}\theta - \cot\theta| + 2\cos\theta) + C_1$$

$\Rightarrow$ The equation of the curve is :

$$y = \pm 2\left(\log\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2}\right)$$

Example-20

Find the equation of the curve passing through the origin if the mid-point of the segment of the normal drawn at any point of the curve and the X-axis lies on the parabola $2y^2 = x$.

Soln. $OB = OM + MB = x + y\tan\theta = x + yy'$

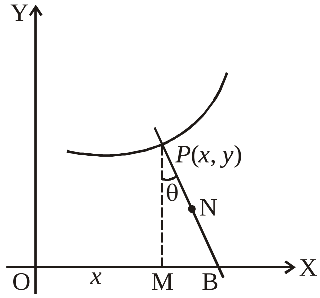
$$\Rightarrow B \equiv (x + yy', 0)$$

$$\Rightarrow N \text{ (mid point of } PB) \equiv \left(x + \frac{yy'}{2}, \frac{y}{2}\right)$$

N lies on $2y^2 = x$

$$\Rightarrow\ 2\left(\frac{y}{2}\right)^2 = x + \frac{yy'}{2} \ \Rightarrow\ yy' - y^2 = -2x$$

(Divide both sides by y and check that it is a Bernouli's differential equation)



Put $y^2 = t \ \Rightarrow\ 2yy' = \frac{dt}{dx}$

$$\Rightarrow\ \frac{1}{2}\frac{dt}{dx} - t = -2x \ \Rightarrow\ \frac{dt}{dx} - 2t = -4x$$

which is a linear differential equation.

I.F. = Integrating factor $= e^{\int -2\,dx} = e^{-2x}$

Using the standard result, the solution of the differential equation is :

$$te^{-2x} = \int -4xe^{-2x}dx$$

$$\Rightarrow\ te^{-2x} = -4\left(\frac{xe^{-2x}}{-2} + \int \frac{e^{-2x}}{2}dx\right)$$

$$\Rightarrow\ te^{-2x} = -4\left\{-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right\} + C$$

$$\Rightarrow\ t = 2x + 1 + Ce^{2x}$$

As it passes through origin ; $C = -1$

$\Rightarrow\ y^2 = 2x + 1 - e^{2x}$ is the required curve.

Example-21

Find equation of curves which intersect the hyperbola $xy = c$ at an angle $\pi/2$.

Soln. Let $m_1 = \frac{dy}{dx}$ for the required family of curves at (x, y).

Let m_2 = value of $\frac{dy}{dx}$ for $xy = c$ curve.

$$m_2 = \frac{dy}{dx} = -\frac{y}{x}$$

As the required family is perpendicular to the given curve, we can have :

$$m_1 \times m_2 = -1$$

$\Rightarrow\ \frac{dy}{dx} \times \left(-\frac{y}{x}\right) = -1 \Rightarrow$ for required family of curves : $\frac{dy}{dx} = \frac{x}{y}$

$$\Rightarrow\ ydy = xdx \ \Rightarrow\ y^2 = x^2 + C_1$$

is the required family which intersects $xy = c$ curve at an angle $\pi/2$.

Concept Application Exercise

1. Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

2. Find the equation of the curve which is such that the area of the rectangle constructed on the abscissa of any point and the intercept of the tangent at this point on the y-axis is equal to 4.

3. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis and the y-axis in point A and B, respectively, such that $\frac{1}{OA} + \frac{1}{OB} = 1$, where O is the origin. Find the equation of such a curve passing through (5, 4).

4. Find the curve such that the intercept on the x-axis cut-off between the origin, and the tangent at a point is twice the abscissa and passes through the point (1, 2).

5. Find the equation of the curve such that the square of the intercept cut-off by any tangent from the y-axis is equal to the product of the coordinates of the point of tangency.

6. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

7. Find the equation of the curve in which the subnormal varies as the square of the ordinate.

8. Find the curve for which the length of normal is equal to the radius vector.

9. Find the curve for which the perpendicular from the foot of the ordinate to the tangent is of constant length.

10. A curve $y = f(x)$ passes through the origin. Through any point (x, y) on the curve, lines are drawn parallel to the co-ordinate axes. If the curve divides the area formed by these lines and co-ordinates axes in the ratio $m : n$, find the curve.