6

Triple Integration & its Applications

1. TRIPLE INTEGRATION

In the preceeding sections we defined and discussed the properties of single and double integrals for functions of single variable and two variables respectively. In this section, we will define triple integrals for functions of three variables.

- The triple integral of $f(x, y, z)$ is denoted by $\iiint_D f(x, y, z)\,dV$ and is defined as

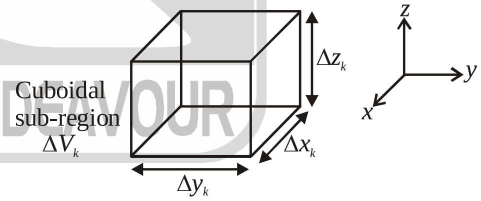
$\iiint_D f(x, y, z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\,dz\right] dA$ where D is a simply xy-solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of D on the xy-plane.

- If D is a rectangular box defined by the inequalities $a \le x \le b,\ c \le y \le d,\ k \le z \le l$, then

$$\iiint_D f(x, y, z)\,dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\,dz\,dy\,dx$$

Volume element

- $dV = dzdydx = dzdxdy = dxdydz = dxdzdy$
  $= dydxdz = dydzdx$
- $\Delta V = (\Delta x)(\Delta y)(\Delta z)$



Evaluating Triple Integrals :

Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G.

The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R.

Step 3. After finding limits of integration, do the integration as in case of double integral i.e., partial integration.

Geometric Interpretation:

From the definition of the triple integral

$$\iiint_D dV = \text{volume of the region D.}$$

Ex.1: Evaluate : $\int_0^1\int_0^2\int_0^3 xyz\,dz\,dx\,dy$

Soln. Here limit of z : $0 \le z \le 3$

Here limit of x : $0 \le x \le 2$

Here limit of y : $0 \le y \le 1$

$$\int_0^1\left[\int_0^2\left(\int_0^3 z\,dz\right)x\,dx\right]y\,dy = \int_0^1\left(\int_0^2 \frac{9}{2}x\,dx\right)y\,dy = \int_0^1 \frac{36}{4}y\,dy = \frac{36}{8}$$

Ex.2: Evalute $\int_{-1}^1\int_0^1\int_1^2 x^2z^2y\,dz\,dx\,dy$

Soln. $\int_{-1}^1\int_0^1\int_1^2 x^2z^2y\,dz\,dx\,dy = \int_{-1}^1 y\,dy\int_0^1 x^2\,dx\int_1^2 z^2\,dz = 0\times\frac{1}{3}\times\frac{7}{3} = 0$

Ex.3: Evalute $\int_{1/3}^{1/2}\int_0^{\pi}\int_0^1 zx\sin xy\,dz\,dy\,dx$

Soln. $\int_{1/3}^{1/2}\int_0^{\pi}\int_0^1 zx\sin xy\,dz\,dy\,dx = \int_{1/3}^{1/2}\int_0^{\pi}\left(\int_0^1 z\,dz\right)x\sin(xy)\,dy\,dx$

$$= \int_{1/3}^{1/2}\int_0^{\pi}\frac{1}{2}x\sin(xy)\,dy\,dx = \frac{1}{2}\int_{1/3}^{1/2}\left(\int_0^{\pi}\sin(xy)\,dy\right)x\,dx$$

$$= \frac{1}{2}\int_{1/3}^{1/2}\left[\frac{-\cos xy}{x}\right]_0^{\pi}x\,dx = \frac{1}{2}\int_{1/3}^{1/2}(1-\cos\pi x)\,dx \text{ ss} = \frac{1}{2}\left[x-\frac{\sin\pi x}{\pi}\right]_{1/3}^{1/2}$$

$$= \frac{1}{2}\left[\frac{1}{2}-\frac{\sin\frac{\pi}{2}}{\pi}-\frac{1}{3}+\frac{\sin\frac{\pi}{3}}{\pi}\right] = \frac{1}{2}\left[\frac{1}{6}+\frac{1}{2\pi}-\frac{1}{\pi}\right] = \frac{1}{2}\left[\frac{1}{6}-\frac{1}{2\pi}\right] = \frac{1}{4}\left(\frac{1}{3}-\frac{1}{\pi}\right)$$

Ex.4: Evalute $\int_0^{\pi/4}\int_0^1\int_0^{x^2} x\cos y\,dz\,dx\,dy$

Soln. $\int_0^{\pi/4}\int_0^1\int_0^{x^2} x\cos y\,dz\,dx\,dy = \int_0^{\pi/4}\int_0^1 [z]_0^{x^2}x\cos y\,dx\,dy = \int_0^{\pi/4}\int_0^1 x^3\cos y\,dx\,dy$

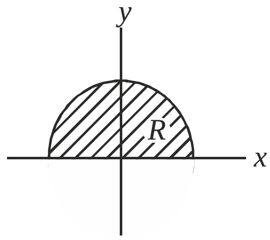
$$= \int_0^{\pi/4}\cos y\,dy\int_0^1 x^3\,dx = \frac{1}{\sqrt{2}}\times\frac{1}{4} = \frac{1}{4\sqrt{2}}$$

Ex.5: Evaluate $\iiint_G y\,dV$, where G is the solid enclosed by the plane z = y, the xy-plane, and the parabolic cylinder $y = 1 - x^2$.

Soln. Limit of z, $0 \le z \le y$

Limit of y, $0 \le y \le 1 - x^2$

Limit of x, $-1 \le x \le 1$



$$\therefore \quad \iiint_G y\,dv = \iint_R \int_0^y y\,dz\,dx\,dy$$

$$= \int_{-1}^{1}\int_0^{1-x^2} y^2\,dy\,dx = \frac{1}{3}\int_{-1}^{1}(1-x^2)^3\,dx$$

$$= \frac{2}{3}\int_{-1}^{1}(1-x^6+3x^4-3x^2)dx = \frac{2}{3}\left(1-\frac{1}{7}+\frac{3}{5}-1\right) = \frac{32}{70}$$

Ex.6: Change the order of the integral $\int_0^9\int_0^{3-\sqrt{x}}\int_0^z f(x,y,z)\,dy\,dz\,dx$

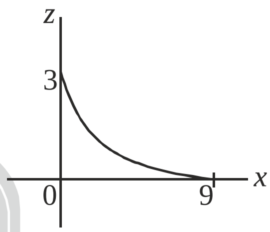
Soln. Limit of x, $0 \le y \le z$

Limit of y, $0 \le z \le 3-\sqrt{x}$

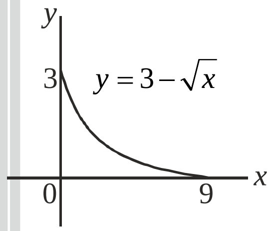
Limit of z, $0 \le x \le 9$

i.e., Region bounded by $y = z$, $z = 3-\sqrt{x}$,

xz-plane, yz-plane



$$\therefore \quad \int_0^9\int_0^{3-\sqrt{x}}\int_0^z f(x,y,z)\,dy\,dz\,dx = \int_0^3\int_0^{(3-z)^2}\int_0^z f(x,y,z)\,dy\,dz\,dx$$



$$= \int_0^9\int_0^{3-\sqrt{x}}\int_y^{3-\sqrt{x}} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^3\int_0^{(y-3)^2}\int_y^{3-\sqrt{x}} f(x,y,z)\,dz\,dy\,dx$$

- **To change the order of integral First we consider**

  (i) region of integral

  (ii) Go parallel to axis suppose we first consider limit of z then we go parallel to z-axis. Put lower limit entering first surface in x, y relation and put upper limit ending with surface in x, y relation.

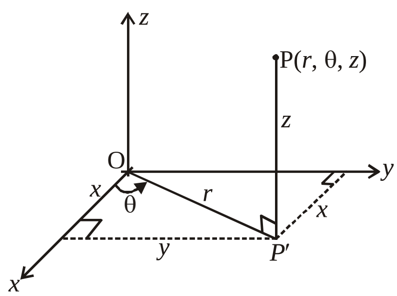
  (iii) Then we find out the relation in x and y with the help of z

  (iv) Now we work just as double integral

1.1 Cylindrical coordinates

Cylindrical coordinates of a point P is (r, θ, z) as shown in the figure. If cartesian coordinates of P be (x, y, z) then

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$
$$r^2 = x^2 + y^2 \text{ and } \tan\theta = y/x$$



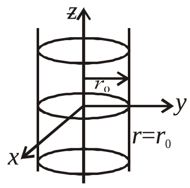
- For one-to-one correspondence from rectangular to cylindrical coordinates we take $r \ge 0$ and $0 \le \theta < 2\pi$.

- Although, there can be six order of integration, similar to the rectangular coordinates. We take only $rdzdrd\theta$ as it gives easier integration.

Simple geometries in cylindrical coordinates:

1. $r = r_0$ (a positive constant)

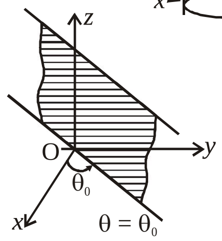
geometry: a right circular cylinder with z-axis its axis and radius as r_0.



2. $\theta = \theta_0 (0 \le \theta_0 \le 2\pi)$ (a constant)

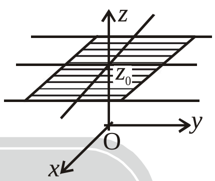
geometry : a plane containing z-axis

and making an angle θ_0 with positive x-axis.

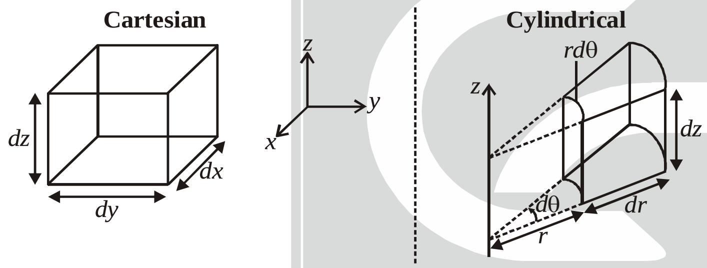


3. $z = z_0$ (a constant)

geometry: a plane perpendicular to the z-axis and cutting z-axis at $(0, 0, z_0)$



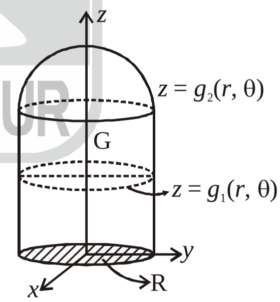
Volume element: Cartesian and cylindrical



$dV = dxdydz$ (Cuboid)

$dV = (rd\theta)(dr)dz = rdzdrd\theta = dzrdrd\theta$ (Cylindrical wedge)

Integration: For the region, G enclosed by $z = g_2(r, \theta)$ and $z = g_1(r, \theta)$ and having projection R in the xy-plane



$$\iiint_G g(r, \theta, z) rdzdrd\theta = \iint_R \left(\int_{g_1(r,\theta)}^{g_2(r,\theta)} g(r, \theta, z) rdz \right) drd\theta$$

G : transformed region in $r\ \theta\ z$ coordinate system.

Transformation to cylindrical coordinates: Transformation from rectangular to cylindrical coordinates.

$$\iiint_D f(x, y, z) dzdxdy = \iiint_G f(r\cos\theta, r\sin\theta, z) rdzdrd\theta$$

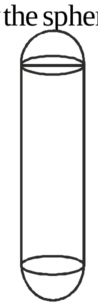
G : transformed region in r,θ,z coordinates

Ex.7: Evalute $\int_0^{\pi/2} \int_0^{\cos\theta} \int_0^{r^2} r \sin\theta\, dzdrd\theta$

Soln. $\int_0^{\pi/2} \int_0^{\cos\theta} \int_0^{r^2} r \sin\theta\, dzdrd\theta = \int_0^{\pi/2} \int_0^{\cos\theta} [z]_0^{r^2} r \sin\theta\, drd\theta$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} r^3 \sin\theta\, dr\, d\theta = \frac{1}{4}\int_0^{\pi/2} \cos^4\theta \sin\theta\, d\theta = \frac{1}{4}\int_0^1 t^4 dt = \frac{1}{20}$$

Ex.8: Find the volume of the solid region that is bounded above and below by the sphere $x^2 + y^2 + z^2 = 9$ and inside the cylinder $x^2 + y^2 = 4$.



Soln. Volume $= \iiint dv$

$$= \int_0^{2\pi}\int_0^2 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\, dz\, dr\, d\theta$$

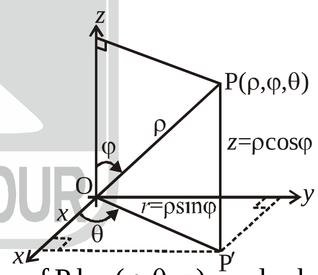
$$= \int_0^{2\pi}\int_0^2 2r\sqrt{9-r^2}\, dr\, d\theta = 2\pi\left[-(9-r^2)^{3/2}\right]_0^2 = 2\pi\left[27-5^{3/2}\right]$$

Ex.9: Evaluate $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\, dz\, dy\, dx$

Soln. $$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\, dz\, dy\, dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2(9-x^2-y^2)\, dy\, dx = \int_0^{2\pi}\int_0^3 x^3\cos^2\theta(9-r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\cos^2 d\theta \int_0^3 (9r^3 - r^5)\, dr = \pi\left(\frac{9}{4}3^4 - \frac{3^6}{6}\right) = \frac{\pi\times 3^5}{4} = \frac{243\pi}{4}$$

Spherical coordinates: Spherical coordinates of a point P is (ρ, φ, θ) as shown in the figure. P' is the perpendicular projection of P on the *xy*-plane.



$|\overrightarrow{OP}| = \rho$

φ = angle between $\overrightarrow{OP}$ and positive *z*-axis

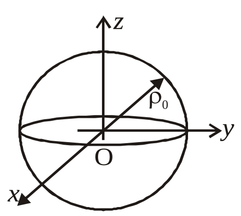
θ = angle between $\overrightarrow{OP'}$ and positive *x*-axis

$0 \le \varphi \le \pi,\ 0 \le \theta < 2\pi$

If cartesian coordinates of P be (x, y, z), cylindrical coordinates of P be (r, θ, z), and spherical coordinates of P be (ρ, φ, θ), then

Rectangular	Cylindrical
$x = \rho\sin\varphi\cos\theta$	$r = \rho\sin\varphi$
$y = \rho\sin\varphi\sin\theta$	$z = \rho\cos\varphi$
$z = \rho\cos\varphi$	$\theta = \theta$

Simple geometries in spherical coordinates:

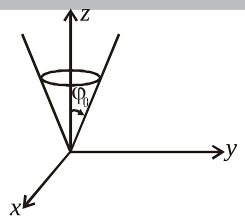


1. $\rho = \rho_0$ (a positive constant)

   geometry: a sphere of radius ρ_0 and centre (0, 0, 0)

2. $\varphi = \varphi_0$ (a constant such that $0 \le \varphi_0 \le \pi$)

   geometry: a right circular single cone with axis z-axis and vertex (0, 0, 0) and half angle φ_0

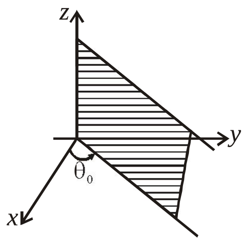


Special case:

$\varphi = 0 \Rightarrow$ positive $z-$axis

$\varphi = \pi/2 \Rightarrow$ xy–plane

$\varphi = \pi \Rightarrow$ negative z-axis

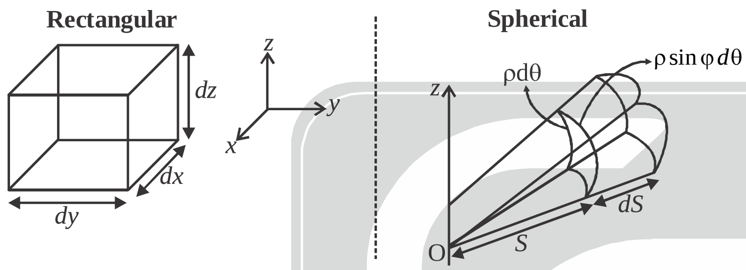


3. $\theta = \theta_0$ (a constant)

geometry: half plane containing z-axis

and making an angle θ_0 with positive x-axis.

Volume elements:



$dV = dxdydz$ (Cuboid) $\quad dV = \rho^2 \sin\varphi ds \cos d\theta = (\rho\sin\varphi d\varphi)(\rho\varphi d\theta)d\rho$ (Spherical wedge)

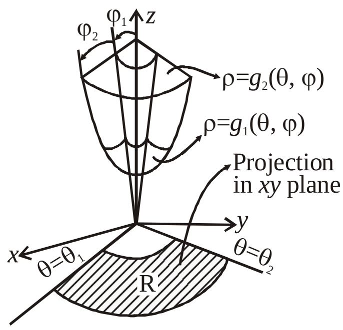
Integration:

$$I = \iiint_D f(x,y,z)dxdydz = \iiint_G f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \rho^2\sin\varphi\, d\rho d\theta d\varphi$$

$$= \iiint_G g(\rho,\varphi,\theta)\rho^2 \sin\varphi d\rho d\theta d\varphi.$$

- Integration in spherical coordinates is useful if G is as

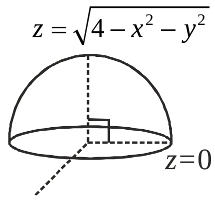
$$G = \{(\rho,\varphi,\theta) : g_1(\varphi,\theta) \le \rho \le g_2(\varphi,\theta), \varphi_1 \le \varphi \le \varphi_2, \varphi_1 \le \varphi \le \varphi_2\}$$



Then, $I = \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} g(\rho,\varphi,\theta)\rho^2 \sin\varphi d\rho d\varphi d\theta \cdot$

Ex. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dzdydx$

Soln. We have,



$0 \le \theta \le 2\pi \qquad x = r\sin\phi\cos\theta$

$0 \le \phi \le \pi/2 \qquad y = r\sin\phi\sin\theta$

$0 \le r \le 2 \qquad z = r\cos\phi$

$$\therefore \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dzdydx$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} r^2\sqrt{r^2}\cos^2\phi r^2\sin\phi drd\phi d\theta = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} r^5\sin\phi\cdot\cos^2\phi drd\phi d\theta$$

$$= 2\pi\cdot\frac{2^6}{6}\int_0^{\pi/2}\sin\phi\cdot\cos^2\phi d\theta = \frac{64\pi}{3}\times\frac{1}{3} = \frac{64\pi}{9}$$

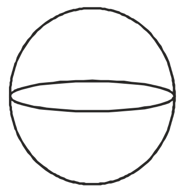
Ex.10: Find the Volume of the sphere of radius a.

Soln. Volume $= \iiint dV$

$$= \iiint dzdxdy$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin\phi drd\phi d\theta = 2\pi\times 2\times\frac{a^3}{a} = \frac{4\pi a^3}{3}$$

$x^2+y^2+z^2=a^2$



Substitutions in Triple Integration

Transforming an integral in (x, y, z) to new coordinates (u, v, w) as

$$x = g(u,v,w),\ y = h(u,v,w) \text{ and } z = k(u,v,w),$$

$$\iiint_D f(x,y,z)dxdydz = \iiint_G f(g(u,v,w),h(u,v,w),k(u,v,w))\,|\,J(u,v,w)\,|\,dudvdw.$$

G: The transformed region in u, v, w coordinate system,

$J(u,v,w)$ is “Jacobian determinant or simply “Jacobian” defined as

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

Note:

1. The g, h, k must be differentiable functions and the transformation must to one-to-one.

2. $J(u,v,w).J(x,y,z)=1$ *i.e.*, $\frac{\partial(x,y,z)}{\partial(u,v,w)}.\frac{\partial(u,v,w)}{\partial(x,y,z)}=1$

3. Usually, the reverse substitution is made, i.e., $u = g_1(x,y,z), v = g_2(x,y,z)$ and $w = g_3(x,y,z)$, and $J(x,y,z)$ is calculated first, then $J(u,v,w) = \frac{1}{J(x,y,z)}$ is evaluated.

4. **Jacobian of transformations to cylindrical coordinates:** $x = r\cos\theta, y = r\sin\theta, z = z$

$$J(r,\theta,z)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r(\geq 0)$$

Hence, $dV = dxdydz = rdzdrd\theta$.

5. Jacobian of transformation to spherical coordinates: $x=\rho\sin\varphi\cos\theta, y=\rho\sin\varphi\sin\theta, z=\rho\cos\varphi$,

$$J(\rho,\varphi,\theta)=\begin{vmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta\\ \cos\varphi & -\rho\sin\varphi & 0\end{vmatrix}=\rho^2\sin\varphi .$$

Ex.11: Let D be the edge in the first octant that is cut from the cylindrical solid $y^2+z^2\leq 1$ by the planes $y = x$ and $x = 0$.

Evaluate $\iiint\limits_G z\,dV$

Soln. According to the figure,

$$\iiint\limits_G z\,dv=\iint\limits_R\left[\int_0^{\sqrt{1-y^2}} z\,dz\right]dA$$

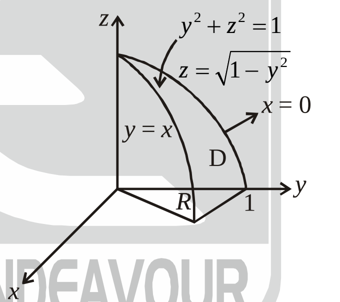
$$=\int_0^1\int_0^y \frac{1}{2}(1-y^2)\,dx\,dy$$

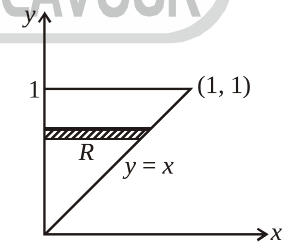
$$=\frac{1}{2}\int_0^1 x(1-y^2)\Big|_0^y\,dy$$

$$=\frac{1}{2}\int_0^1 y(1-y^2)\,dy$$

$$=\frac{1}{2}\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1$$

$$=\frac{1}{8}\ \textbf{Ans.}$$





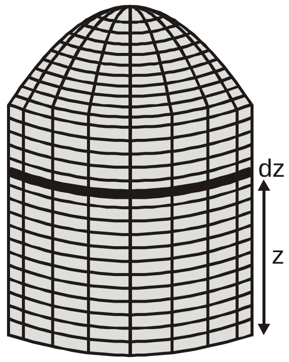
2. VOLUME OF SOLID

2.1 Volume in Cartesian coordinates

The volume of a solid can be found in the following 3-ways:

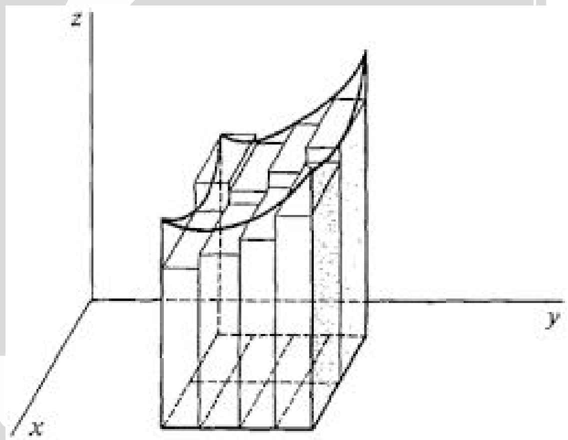
Volume by Single Integration : We suppose the solid to be cut by a plane perpendicular to the line chosen (say, the z-axis) at a distance z from the origin. The section of the solids thus obtained is a plane section whose area is a function of z, say, $f(z)$ and then consider an another plane at a distance $z + dz$. The volume of the solid between these planes is clearly $f(z)dz$.

Therefore the volume of the solid $= \int_a^b f(z)\,dz$. This method is applicable where slice of volume at height *z* is having regular figures like circle, square, rectangle and triangle etc. Area of these slices can be obtained as a function of *z*.

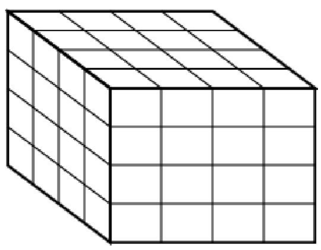


Volume by Double Integration : Consider the plane section of the solid by the plane $z = 0$. Now an element of the area on the plane $z = 0$ is *dxdy*. We now consider a cylinder, formed by the area *dxdy*, the lines parallel to *z*-axis and the area of the surface (given) cut by these lines. The volume of this elementary cylinder formed is $z\,dx\,dy$.

Hence, the volume of the solid $= \iint z\,dx\,dy$.



Volume by Triple Integration : Taking the elementary volume (i.e. volume of the small element) *dx dy dz*, the volume of the solid is $\iiint dx\,dy\,dz$, where the integration is performed under suitable limits such as to include the whole volume.



9.11 Triple Integral (Notation)

Functions of three variable : $f(x, y, z), g(x, y, z), ...$

Triple integrals : $\iiint\limits_G f(x,y,z)\,dV, \iiint\limits_G g(x,y,z)\,dV, ...$

Riemann sum : $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_1 \Delta y_j\, \Delta z_k$

Small changes : $\Delta x_i, \Delta y_j, \Delta z_k$

Limit of integration: a, b; c, d; r, s

Regions of integration : G, T, S

Cylindrical coordinates: r, θ, z

Spherical coordinates: r, θ, φ

Volume of a solid : V

Mass of a solid : m

Density : $\mu(x,y,z)$

Coordinates of centre of mass: $\bar{x}, \bar{y}, \bar{z}$

First momemts : M_{xy}, M_{yz}, M_{xz}

Moments of inertia: $I_{xy},\ I_{yz}, I_{xz}, I_x, I_y, I_z, I_0$

17. Definition of Triple Integral

The triple integral over a parallelepiped $[a,b]\times[c,d]\times[r,s]$ is defined to be

$$\iiint\limits_{[a,b]\times[c,d]\times[r,s]} f(x,y,z)\,dV = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_i \to 0 \\ \max \Delta z_k \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_i \Delta z_k,$$

Where (u_i, v_j, w_k) is some point in the parallelepiped

$(x_{i-1}, x_i)\times(y_{j-1}, y_j)\times(z_{k-1}, z_k)$, and $\Delta x_i = x_i - x_{i-1}$,

$\Delta y_j = y_j - y_{j-1}, \Delta z_k = z_k - z_{k-1}$

18. $\iiint\limits_G [f(x,y,z)+g(x,y,z)]\,dV = \iiint\limits_G f(x,y,z)\,dV + \iiint\limits_G f(x,y,z)\,dV$

19. $\iiint\limits_G [f(x,y,z)-g(x,y,z)]\,dV = \iiint\limits_G f(x,y,z)\,dV - \iiint\limits_G g(x,y,z)\,dV$

20. $\iiint\limits_G kf(x,y,z)\,dV = k\iiint\limits_G f(x,y,z)\,dV$ where is k is a constant.

21. If $f(x,y,z) \ge 0$ and G and T are non overlapping basic regions, then

$$\iiint_{G\cup T} f(x,y,z)\,dV = \iiint_G f(x,y,z)\,dV + \iiint_T f(x,y,z)\,dV.$$

Here $G \cup T$ is the union of the regions G and T.

22. Evalution of Triple Integrals by Repeated Integrals

If the solid G is the set of points (x, y, z) such that

$(x,y)\in R, \chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint_G f(x,y,z)\,dxdydz = \iint_R \left[\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,dz\right] dxdy,$$

Where R is projection of G onto the xy-plane.

If the solid G is the set of points (x, y, z) such that

$a\le x\le b, \varphi_1(x)\le y\le \varphi_2(x), \chi_1(x,y)\le z\le x_2(x,y)$, then

$$\iiint_G f(x,y,z)\,dxdydz = \int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,dz\right)dy\right]dx$$

23. Triple Integrals over Parallelepiped $[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_G f(x,y,z)\,dxdydz = \int_a^b\left[\int_c^d\left(\int_r^s f(x,y,z)\,dz\right)dy\right]dx$$

In the special case where the integral f(x, y, z) can be written as g(x) h (y) k (z) we have

$$\iiint_G f(x,y,z)\,dxdydz = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)\left(\int_r^s k(z)\,dz\right)$$

24. **Change of Variables**

$$\iiint_G f(x,y,z)\,dxdydz = \iiint_G f\left[x(u,v,w)\,y,(u,v,w),z(u,v,w)\right]\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dxdydz,$$

Where $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$ is the jacobian of the transformations

$(x,y,z)\to(u,v,w)$, and S is the pull back of G which can be computed by $x = x(u,v,w)$

$y = y(u,v,w)\;\; z = z(u,v,w)$ into the definition of G .

25. Triple Integrals in Cylindrical Coordinates the differential dxdydz for cylinderical coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| drd\theta dz = rdrd\theta dz$$

Let the solid G is determined as follows:

$$(x,y)\in, \chi_1(x,y)\le z\le \chi_2(x,y),$$

Where R is projection of G on to the xy- plane. Then

$$\iiint_G f(x,y,z)\,dx\,dydz = \iiint_S f(r\cos\theta, r\sin\theta, z)\,rdrd\theta dz = \iint_{R(r,\theta)}\left[\int_{\chi_1(r\cos\theta, r\sin\theta)}^{\chi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\right] rdrd\theta$$

Here S is the pullback of G in cylindrical coordinates.

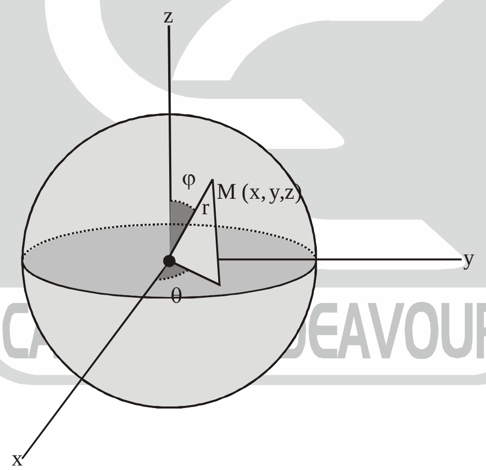
26. **Triple Integrals in Spherical Coordinates**

The Differential dxdydz for Spherical Coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| drd\theta d\varphi = r^2\sin\theta drd\theta d\varphi$$

$$\iiint_G f(x,y,z)\,dxdydz = \iiint_S f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\,r^2\sin\theta\,drd\theta d\varphi,$$

Where the solid S is the pullback of G in spherical coordinates. The angle θ ranges from 0 to 2π, the angle φ rantges from 0 to π.



27. Volume of S solid in cartesian coordinates

$$V = \iiint_G dxdydz$$

28. Volume in Cylindrical Coordinates

$$V = \iiint_{s(r,\theta,z)} rdrd\theta dz$$

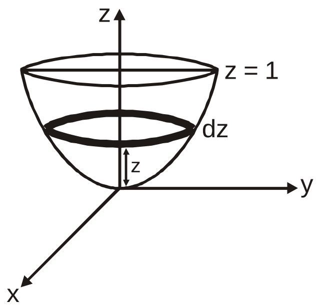
29. Volume in sperical Coordinates

$$V = \iiint_{S(s,\theta,\varphi)} r^2\sin\theta\,drd\theta\,d\varphi$$

Ex.13: Find the volume bounded by $z = x^2 + y^2$ and $z = 1$

Soln. This question can be solved by single integration, double integration as well as by triple integration

By Single Integration

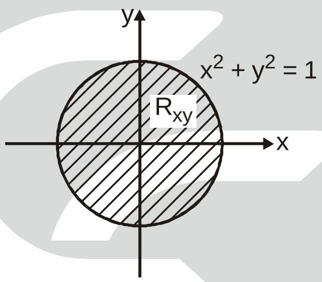


$$V = \int_{z=0}^{1} f(z)\,dz = \int_{z=0}^{1} \pi\left(\sqrt{z}\right)^2 dz = \pi\left|\frac{z^2}{2}\right|_0^1 = \frac{\pi}{2}.$$

By Double Integration

$$V = \iint_{R_{xy}} \left(1-(x^2+y^2)\right) dx\,dy$$

R_{xy} is projection of volume at *xy* plane



Solving in polar coordinate system

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1-r^2)\,r\,dr\,d\theta = \left|\frac{r^2}{2}-\frac{r^4}{4}\right|_0^1 2\pi = 2\pi\cdot\left(\frac{1}{2}-\frac{1}{4}\right) = 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}.$$

By Triple Integration

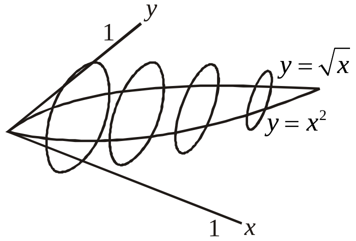
$$V = \iint_{R_{xy}}\int_{z=x^2+y^2}^{z=1} dz\,dy\,dx = \iint_{R_{xy}}\left(1-(x^2+y^2)\right)dx\,dy = \frac{\pi}{2}$$

Thus, volume can be calculated by any method. Here first method can be applied because slice of volume is of circular shape.

Ex.14: The solid lies between planes perpendicular to the *x*-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the *x*-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

Soln. Area of disc formed at general *x* is $A(x)$

$$A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{x}-x^2\right)^2 = \frac{\pi}{4}\left(x - 2\sqrt{x}\cdot x^2 + x^4\right);\ a = 0, b = 1$$

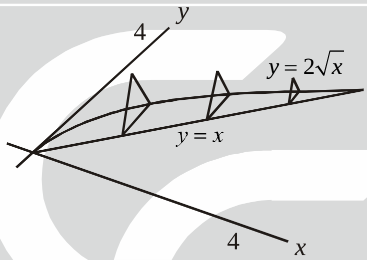


$$\Rightarrow\ V=\int_a^b A(x)\,dx=\frac{\pi}{4}\int_0^1\left(x-2x^{5/2}+x^4\right)dx=\frac{\pi}{4}\left[\frac{x^2}{2}-\frac{4}{7}x^{7/2}+\frac{x^5}{5}\right]_0^1=\frac{\pi}{4}\left(\frac{1}{2}-\frac{4}{7}+\frac{1}{5}\right)$$

$$=\frac{\pi}{4\cdot 70}(35-40+14)=\frac{9\pi}{280}.$$

Ex.15: The base of the solid is the region in the first quadrant between the line $y=x$ and the parabola $y=2\sqrt{x}$. The cross sections of the solid perpendicular to the *x*-axis are equilateral triangles whose bases stretch from the line to the curve.

Soln. $A(x)=\frac{1}{2}(\text{side})^2\left(\sin\frac{\pi}{3}\right)=\frac{\sqrt{3}}{4}\left(2\sqrt{x}-x\right)^2=\frac{\sqrt{3}}{4}\left(4x-4x\sqrt{x}+x^2\right);\ a=0,\ b=4$



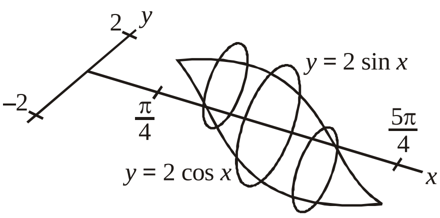
$$\Rightarrow\ V=\int_a^b A(x)\,dx=\frac{\sqrt{3}}{4}\int_0^4\left(4x-4x^{3/2}+x^2\right)dx=\frac{\sqrt{3}}{4}\left[2x^2-\frac{8}{5}x^{5/2}+\frac{x^3}{3}\right]_0^4=\frac{\sqrt{3}}{4}\left(32-\frac{8\cdot 32}{5}+\frac{64}{3}\right)$$

$$=\frac{32\sqrt{3}}{4}\left(1-\frac{8}{5}+\frac{2}{3}\right)=\frac{8\sqrt{3}}{15}(15-24+10)=\frac{8\sqrt{3}}{15}.$$

Ex.16: The solid lies between planes perpendicular to the *x*-axis at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$. The cross-sections between these planes are circular disks whose diameters run from the curve $y=2\cos x$ to the curve $y=2\sin x$.

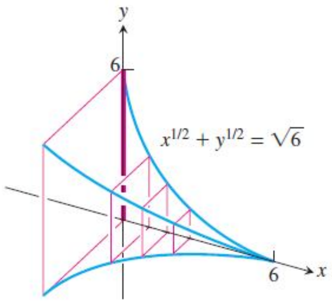
Soln. $A(x)=\frac{\pi}{4}(\text{diameter})^2=\frac{\pi}{4}(2\sin x-2\cos x)^2=\frac{\pi}{4}\cdot 4(\sin^2 x-2\sin x\cos x+\cos^2 x)$

$$=\pi(1-\sin 2x);\ a=\frac{\pi}{4},\ b=\frac{5\pi}{4}.$$



$$\Rightarrow\ V=\int_a^b A(x)\,dx=\pi\int_{\pi/4}^{5\pi/4}(1-\sin 2x)\,dx=\pi\left[x+\frac{\cos 2x}{2}\right]_{\pi/4}^{5\pi/4}=\pi\left[\left(\frac{5\pi}{4}+\frac{\cos\frac{5\pi}{2}}{2}\right)-\left(\frac{\pi}{4}-\frac{\cos\frac{\pi}{2}}{2}\right)\right]=\pi^2.$$

Ex.17: The solid lies between planes perpendicular to the x-axis at $x=0$ and $x=6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2}+y^{1/2}=\sqrt{6}$.



Soln. $A(x)=(\text{edge})^2=\left(\left(\sqrt{6}-\sqrt{x}\right)^2-0\right)^2=\left(\sqrt{6}-\sqrt{x}\right)^4=36-24\sqrt{6}\sqrt{x}+36x-4\sqrt{6}x^{3/2}+x^2$;

$$a=0,\ b=6\Rightarrow V=\int_a^b A(x)dx=\int_0^6\left(36-24\sqrt{6}\sqrt{x}+36x-4\sqrt{6}x^{3/2}+x^2\right)dx$$

$$=\left[36x-24\sqrt{6}\cdot\frac{2}{3}x^{3/2}+18x^2-4\sqrt{6}\cdot\frac{2}{5}x^{5/2}+\frac{x^3}{3}\right]_0^6=216-16\cdot\sqrt{6}\sqrt{6}\cdot 6+18\cdot 6^2-\frac{8}{5}\sqrt{6}\sqrt{6}\cdot 6^2+\frac{6^3}{3}$$

$$=216-576+648-\frac{1728}{5}+72=360-\frac{1728}{5}=\frac{1800-1728}{5}=\frac{72}{5}.$$

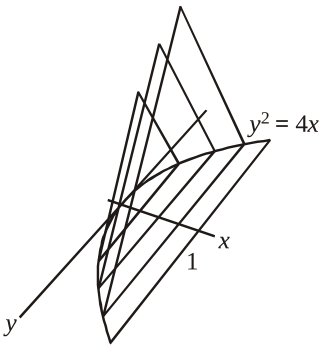
Ex.18: The solid lies between planes perpendicular to the x-axis at $x=0$ and $x=4$. The cross sections of the solid perpendicular to the x-axis between these planes are circular disks whose diameters run from the curve $x^2=4y$ to the curve $y^2=4x$.

Soln. $A(x)=\frac{\pi}{4}(\text{diameter})^2=\frac{\pi}{4}\left(2\sqrt{x}-\frac{x^2}{4}\right)^2=\frac{\pi}{4}\left(4x-x^{5/2}+\frac{x^4}{16}\right)$; $a=0,\ b=4\Rightarrow V=\int_a^b A(x)\,dx$

$$=\frac{\pi}{4}\int_0^4\left(4x-x^{5/2}+\frac{x^4}{16}\right)dx=\frac{\pi}{4}\left[2x^2-\frac{2}{7}x^{7/2}+\frac{x^5}{5\cdot 16}\right]_0^4=\frac{\pi}{4}\left(32-32\cdot\frac{8}{7}+\frac{2}{5}\cdot 32\right)=\frac{72\pi}{35}$$

Ex.19: The base of the solid is the region bounded by the parabola $y^2=4x$ and the line $x=1$ in the xy-plane. Each cross-section perpendicular to the x-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane).

Soln. $A(x)=\frac{1}{2}(\text{edge})^2\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{4}\left[2\sqrt{x}-\left(-2\sqrt{x}\right)\right]^2=\frac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2=4\sqrt{3}x$; $a=0,\ b=1$



$$\Rightarrow\ V=\int_a^b A(x)\,dx=\int_0^1 4\sqrt{3}x\,dx=\left[2\sqrt{3}x^2\right]_0^1=2\sqrt{3}.$$

5. DIRICHLET'S THEOREM

Statement : The theorem states that :

$$\iiint \ldots \int x_1^{m_1-1} x_2^{m_2-1} x_3^{m_3-1}, \ldots x_n^{m_n-1}\, dx_1\, dx_2\, dx_3 \ldots dx_n = \frac{\Gamma(m_1)\,\Gamma(m_2)\,\Gamma(m_3)\ldots\Gamma(m_n)}{\Gamma(m_1+m_2+\cdots+m_n+1)},$$

where the integral is extended to all positive values of the variables $x_1, x_2, \ldots, x_n$ subject to the condition $x_1 + x_2 + \cdots + x_n \le 1$.

LIOUVILLE'S EXTENSION OF DIRICHLET'S THEOREM

Statement : If x, y, z are all positive such that $h_1 \le x + y + z \le h_2$

$$\iiint F(x+y+z)\cdot x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(u)\, u^{l+m+n-1}\, du$$

Cor. 1. $I = \iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \displaystyle\int_{h_1}^{h_2} F(u)\cdot u^{l+m+n-1}\, du$

when the integral *I* is extended to all positive values of the variables x, y, z subject to the condition $h_1 < x + y + z < h_2$.

Cor. 2. General form.

$$\iint \ldots \int F(t_1 + t_2 + \cdots + t_n)\, t_1^{m_1-1} t_2^{m_2-1} \ldots t_n^{m_n-1}\, dt_1\, dt_2 \ldots dt_n = \frac{\overline{|m_1}\,\overline{|m_2}\ldots\overline{|m_n}}{\overline{|m_1+m_2+\cdots+m_n}} \int_{h_1}^{h_2} t^{m_1+m_2+\cdots m_n-1}\, dt$$

can be reduced to a simple, where *F* is continuous, $m_r > 0\ (r = 1, 2, \ldots, n)$ and the integration is extended over all positive values of the variable such that $h_1 \le t_1 + t_2 + \cdots + t_n \le h_2$.
It is Liouville's extension for *n* variables.

Ex.20: Evaluate $\iiint x\, y\, z\, dx\, dy\, dz$ taking throughout the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$.

Soln. Let $\dfrac{x^2}{a^2} = u,\ x = au^{1/2},\ dx = \dfrac{1}{2} au^{-1/2}\, du$

$\dfrac{y^2}{b^2} = v,\ y = bv^{1/2},\ dy = \dfrac{1}{2} bv^{-1/2}\, dv$

$\dfrac{z^2}{c^2} = w,\ z = cw^{1/2},\ dz = \dfrac{1}{2} cw^{-1/2}\, dw$

Therefore, when the given integral is taken for positive values of x, y, z in positive octant, the given integral reduces to $= \dfrac{a^2b^2c^2}{8} \iiint du\, dv\, dw$.

Subject to the condition $u + v + w \le 1$

$$= \frac{a^2b^2c^2}{8} \iiint u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw = \frac{a^2b^2c^2}{8} \frac{(\Gamma 1)^3}{\Gamma(1+1+1+1)} \quad \text{[By Dirichlet's Theorem]}$$

$$=\frac{a^2b^2c^2}{8}\cdot\frac{1}{\Gamma(4)}=\frac{a^2b^2c^2}{8\times 6}$$

Hence, for the whole ellipsoid, the given integral is

$$=8\cdot\frac{a^2b^2c^2}{8\times 6}=\frac{a^2b^2c^2}{6}.$$

Ex.21: Evaluate $\iiint x^{-1/2}\,y^{-1/2}\,z^{-1/2}(1-x-y-z)^{1/2}\,dx\,dy\,dz$ extended to all positive values of the variables subject to the condition $x+y+z<1$.

Soln. When the condition is $0<x+y+z<1$, we have the given integral

$$=\iiint x^{\frac{1}{2}-1}\,y^{\frac{1}{2}-1}\,z^{\frac{1}{2}-1}\left\{1-(x+y+z)\right\}^{1/2}dx\,dy\,dz=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}\int_0^1(1-u)^{1/2}\,u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,du$$

$$=\frac{\left(\sqrt{\pi}\right)^3}{\Gamma\left(\frac{3}{2}\right)}\int_0^1(1-u)^{1/2}\,u^{1/2}\,du=\frac{\left(\sqrt{\pi}\right)^3}{\frac{1}{2}\Gamma\frac{1}{2}}\int_0^1(1-u)^{\frac{3}{2}-1}\,u^{\frac{3}{2}-1}\,du$$

$$=\frac{\left(\sqrt{\pi}\right)^3}{\frac{1}{2}\sqrt{\pi}}\;\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}$$ [Using Liouville's extension of Dirichlet's Theorem]

$$=2\pi\cdot\frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2.1.1}=\frac{\pi^2}{4}.$$

Ex.22: Evaluate : $\iiint e^{x+y+z}dx\,dy\,dz$ taken over positive octant such that $x+y+z\le 1$.

Soln. Here, the condition is $0<x+y+z\le 1$. Hence the given integral

$$=\iiint e^{x+y+z}\cdot x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz=\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(3)}\int_0^1 e^u\,u^{3-1}\,du$$

$$=\frac{1}{2}\left[\left\{u^2.e^2\right\}_0^1-\int_0^1 2u\,.e^u\,du\right]=\frac{1}{2}\left[e-2\left\{\left(e^u.u\right)_0^1-\int_0^1 1.e^u\,du\right\}\right]$$

$$=\frac{1}{2}\left[e-2e+2\left\{e^u\right\}_0^1\right]=\frac{1}{2}\left[e-2e+2e-2\right]=\frac{1}{2}(e-2).$$

Ex.23: The value of the integral $\iiint_D\left(a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2\right)^{1/2}dxdydz$ where D is the region $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1,$ is **[CUCET-2016]**

(a) $a^2b^2c^2\pi^2$ (b) $\frac{1}{4}a^2b^2c^2\pi^2$ (c) $\frac{1}{3}a^2b^2c^2\pi^2$ (d) $\frac{1}{2}a^2b^2c^2\pi^2$

Soln. $\iiint_D (a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2)^{1/2}\,dxdydz = abc\iiint\left(1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right)^{1/2} dxdydz$

Put $\frac{x}{a}=u,\quad \frac{y}{b}=v,\quad \frac{z}{c}=w$

$\therefore\ dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw = abc\,dudvdw$

and $D:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \quad \Rightarrow S: u^2+v^2+w^2\le 1$

Therefore,

$$= a^2b^2c^2\iiint (1-(u^2+v^2+w^2))^{1/2}\,dudvdw$$

In polar form

$$= a^2b^2c^2\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2(1-r^2)^{1/2}\sin\phi\, dr\, d\phi\, d\theta = 2\pi a^2b^2c^2\int_0^{\pi}\sin\phi\, d\phi\int_0^1 r^2(1-r^2)^{1/2}dr$$

$$= 4\pi a^2b^2c^2\int_0^{\pi/2}\sin^2\theta\cdot\cos^2\theta\, d\theta = 4\pi a^2b^2c^2\times\frac{\left|\frac{3}{2}\right.\left|\frac{3}{2}\right.}{2\left|3\right.} = 4\pi a^2b^2c^2\times\frac{1}{4}\times\frac{\pi}{2\times 2!} = \frac{a^2b^2c^2\pi}{4}$$

Ex.24: $\iiint_V x^2y\,dxdydz$, where $V: x^2+y^2\le 1,\ 0\le z\le 1$, is **[ISM-2015]**

(a) $4\pi/15$ (b) $\pi/5$ (c) 0 (d) $2\pi/15$

Soln. $\iiint_V x^2y\,dxdydz = \iint_R\int_0^1 x^2y\,dzdxdy \quad \{R: x^2+y^2\le 1\}$

$$= \iint_R x^2y[z]_0^1\,dxdy = \iint_R x^2y\,dxdy = \int_0^{2\pi}\int_0^1 r^4\cos^2\theta.\sin\theta\, dr\, d\theta = \frac{1}{5}\int_0^{2\pi}\cos^2\theta\cdot\sin\theta\, d\theta = 0$$

Ex.25: The volume of the solid which is bounded by the cylinder $x^2+y^2=1$ and the planes $y+z=1$ and $z=0$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π **[ISM-2015]**

Soln. $V=\iiint dV$

$$= \iint_R\int_0^{1-y} dzdxdy = \iint_R [z]_0^{1-y}dxdy = \iint_R (1-y)dxdy \qquad \{R: x^2+y^2\le 1\}$$

In polar form

$$= \int_0^{2\pi}\int_0^1 (1-r\sin\theta)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta - \int_0^{2\pi}\int_0^1 r^2\sin\theta\, d\theta = 2\pi\times\frac{1}{2}-\frac{1}{3}\times 0 = \pi$$

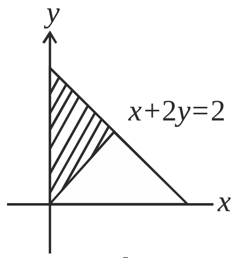
Ex.26: The volume of solid bounded by the planes $x+2y+z=2,\ x=2y,\ x=0$ and $z=0$ is

(a) $\int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} dz\,dx\,dy$

(b) $\int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} dzdydx$ **[JAM(CA)-2010]**

(c) $\int_0^1 \int_0^{2y} \int_0^{2-x-2y} dzdydx$

(d) $\int_0^1 \int_0^{1/2} \int_0^{2-x-2y} dzdydx$

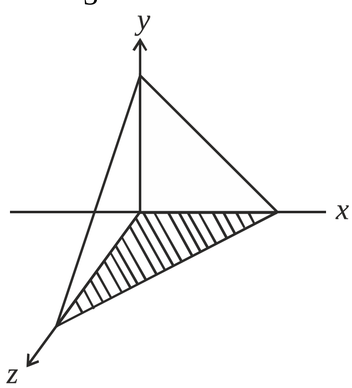


Soln. Volume $=\iiint dV$

$$=\int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dzdydx$$

Ex.27: The volume of the closed region bounded by the planes $x=0, y=0, z=0$ and $2x+5y+10z=10$ is

(a) $\frac{20}{3}$ (b) 5 (c) $\frac{10}{3}$ (d) $\frac{5}{3}$ **[JAM(CA)-2008]**



Soln. Volume $=\iiint dV$

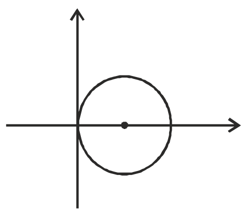
$$=\int_0^5 \int_0^{2-\frac{2x}{5}} \int_0^{1-\frac{2x+5y}{10}} dzdydx$$

$$=\int_0^5 \int_0^{2-\frac{2x}{5}} \left(1-\frac{2x+5y}{10}\right) dydx$$

$$=\int_0^5 \left[\left(1-\frac{x}{5}\right)y-\frac{y^2}{4}\right]_0^{2-\frac{2x}{5}} dx = \int_0^5 \left[\left(1-\frac{x}{5}\right)\left(2-\frac{2x}{5}\right)-\frac{1}{4}\left(2-\frac{2x}{5}\right)^2\right] dx = \frac{5}{3}$$

Ex.28: The volume of the closed region bounded by the surfaces $x^2+y^2=2x,\ z=-1$ and $z=1$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) 2π (d) π **[JAM(CA)-2011]**



Soln. Volume $=\iiint dV = \iint_R \int_{-1}^{1} dzdxdy = \iint_R 2dxdy$

In polar form

$x^2+y^2=2x \Rightarrow R=2\cos\theta,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

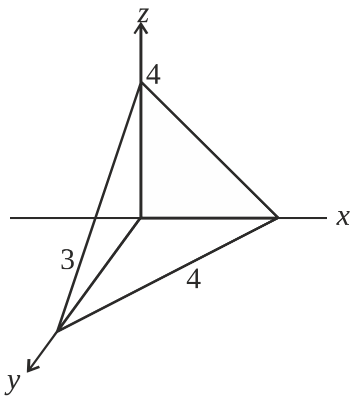
$$=2\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} rdrd\theta = 2\int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = 2\int_{-\pi/2}^{\pi/2} 2\cos^2\theta d\theta$$

$$=8\int_0^{\pi/2} \cos^2\theta d\theta = 8\times\frac{\pi}{4}=2\pi$$

Ex.29: The value of the region in R^3 given by $3|x|+4|y|+3|z|\le 12$ is

(a) 64 (b) 48 (c) 32 (d) 24 **[JAM(CA)-2011]**

Soln. Volume $=8\times\iiint dV$



$$=8\times\int_0^4\int_0^{3-\frac{3x}{4}}\int_0^{4-x-4y/3} dzdydx = 8\times\int_0^4\int_0^{3-\frac{3x}{4}}\left(4-x-\frac{4y}{3}\right)dydx$$

$$=8\times\int_0^4\left[(4-x)y-\frac{2y^2}{3}\right]_0^{3-\frac{3x}{4}} dx$$

$$=8\times\int_0^4\left[(4-x)\left(3-\frac{3x}{4}\right)-\frac{2}{3}\left(3-\frac{3x}{4}\right)^2\right]dx = 8\times 8 = 64$$

Alternative solution

for $x\ge 0, y\ge 0, z\ge 0$

$3x+4y+3z\le 12$

$\Rightarrow \frac{x}{4}+\frac{y}{3}+\frac{z}{4}\le 1$

$\therefore$ volume $=\frac{1}{6}\times 4\times 3\times 4 = 8$

$\therefore$ Volume of region $3|x| + 4|y| + 3|2| \le 2 = 8\times 8 = 64$

Ex.30: The volume of the tetrahedron bounded by the planes $z=0, x=0, y=0$ and $y+z-x=1$ is

(a) $\frac{1}{6}$ (b) 6 (c) 1 (d) $\frac{1}{3}$ **[JAM(CA)-2012]**

Soln. Volume $=\iiint dV = \int_{-1}^{0}\int_0^{1+x}\int_0^{1+x-y} dzdydx = \int_{-1}^{0}\int_0^{1+x}(1+x-y)dydx$

$$=\int_{-1}^{0}\left[(1+x)y-\frac{y^2}{2}\right]_0^{1+x} dx = \int_{-1}^{0}\left((1+x)^2-\frac{(1+x)^2}{2}\right)dx = \frac{1}{2}\int_{-1}^{0}(1+x)^2 dx = \frac{1}{2}\left[\frac{(1+x)^3}{3}\right]_{-1}^{0} = \frac{1}{6}$$

Ex.31: The volume of the region bounded by the surfaces $z=4-\sqrt{x^2+y^2}$ and $z=\sqrt{x^2+y^2}$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{8\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{16\pi}{3}$ **[JAM(CA)-2013]**

Soln. Volume $=\iiint dV = \iint_R \int_{\sqrt{x^2+y^2}}^{4-\sqrt{x^2+y^2}} dzdA = \iint_R\left(4-\sqrt{x^2+y^2}-\sqrt{x^2+y^2}\right)dA = \iint_R\left(4-2\sqrt{x^2+y^2}\right)dxdy$

$$4-\sqrt{x^2+y^2}=\sqrt{x^2+y^2}\Rightarrow x^2+y^2=4$$

$\therefore$ R : $x^2+y^2\le 4$

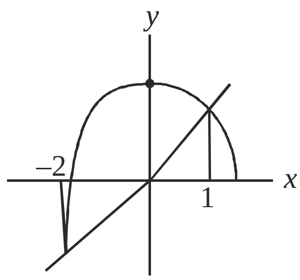
$\therefore$ Volume $= \int_0^{2\pi}\int_0^2 (4-2r) r\,dr\,d\theta = 2\pi\int_0^2 (4r-2r^2)dr = \frac{16\pi}{3}$

Ex.32: Find the volume of the solid whose base is the region in the *xy*-plane that is bounded by the parabola $y = 2-x^2$ and the line $y = x$, while the top of the solid is bounded by the plane $z = x+2$.

[JAM(MS)-2007]

Soln. Volume $= \iiint dv$

$$= \int_{-2}^{1}\int_{x}^{2-x^2}\int_{0}^{x+2} dz\,dy\,dx$$



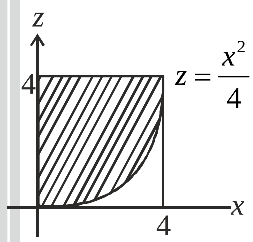
Point of intersection :

$$2-x^2 = x \Rightarrow x^2+x-2=0 \Rightarrow x=-2,1$$

$$= \int_{-2}^{1}\int_{x}^{2-x^2}(x+2)dy\,dx = \int_{-2}^{1}(x+2)(2-x^2-x)dx = 6.75$$

Ex.33: Evaluate the triple integral : $\int_{z=0}^{z=4}\int_{x=0}^{x=2\sqrt{z}}\int_{y=0}^{y=\sqrt{4z-x^2}} dy\,dx\,dz$ __________ **[JAM(MS)-2012]**

Soln. $= \int_{z=0}^{z=4}\int_{x=0}^{x=2\sqrt{z}}\int_{y=0}^{y=\sqrt{4z-x^2}} dy\,dx\,dz = \int_0^4\int_0^{2\sqrt{z}}\sqrt{4z-x^2}\,dx\,dz$



Change the order of integration.

$$= \int_0^4\int_{x^2/4}^{4}\sqrt{4z-x^2}\,dx\,dz$$

$$= \int_0^4\left[\frac{(4z-x^2)^{3/2}}{\frac{3}{2}\times 4}\right]_{x^2/4}^{4} = \int_0^4\left(\frac{(16-x^2)^{3/2}}{6} - \frac{\left(\frac{4x^2}{4}-x^2\right)^{3/2}}{6}\right)dx = \frac{1}{6}\int_0^4(16-x^2)^{3/2}dx$$

Put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}}(16-16\sin^2\theta)^{3/2}\cdot 4\cos\theta d\theta = \frac{4}{6}\times(16)^{3/2}\int_0^{\frac{\pi}{2}}\cos^3\theta\cdot\cos\theta d\theta$$

$$= \frac{128}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta = \frac{128}{3}\times\frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)} = \frac{64}{3}\times\frac{\frac{3}{2}\times\frac{1}{2}\times\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(3)} = \frac{64}{3}\times\frac{3}{4\times 2}\times\pi = 8\pi$$

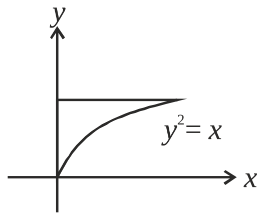
Ex.34: Let m be a real number such that $m > 1$. If $\int_1^m \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\,dx\,dz = e-1$, then m = ______

[JAM(MS)-2016]

Sol. $$\int_1^m \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dydxdz = \int_1^m \int_0^1 \int_0^{y^2} e^{y^3}\, dxdydz$$

By change in order of x & y

$$= \int_1^m \int_0^1 y^2 e^{y^3}\, dydz = \frac{1}{3}\int_1^m \left[e^{y^3}\right]_0^1 dz = \frac{1}{3}\int_1^m (e-1)dz = \frac{1}{3}(m-1)(e-1)$$



$$\Rightarrow \frac{1}{3}(m-1)(e-1) = e-1 \Rightarrow m = 4$$

EXERCISE-1

PART-A (Multiple Choice Questions (MCQ))

1. The value of $\int_{2}^{3}\int_{-1}^{4}\int_{-1}^{0} 4x^2y - z^3 dzdydx$ is

(a) $\left(\frac{-755}{4}\right)$ (b) $\left(\frac{755}{2}\right)$

(c) $\left(\frac{755}{4}\right)$ (d) None of these

2. The value of $\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)dxdydz$ is

(a) 8 (b) –8 (c) 10 (d) 4

3. The value of $\int_{0}^{1}\int_{0}^{2}\int_{0}^{3} dzdydx$ is

(a) 8 (b) 6 (c) 12 (d) 18

4. The value of $\int_{0}^{a}\int_{0}^{b}\int_{0}^{c}(x+y+z)dzdydx$ is

(a) $\left(\frac{abc(a-b+c)}{2}\right)$ (b) $\left(abc(a+b+c)\right)$

(c) $\left(\frac{abc(a+b+c)}{2}\right)$ (d) None of these

5. c $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dydxdz$ is

(a) 2 (b) 3 **(c)** 0 (d) 4

6. The value of $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-z} xyzdzdydx$ is

(a) $\left(\frac{1}{360}\right)$ (b) $\left(\frac{1}{720}\right)$

(c) $\left(\frac{1}{620}\right)$ (d) None of these

7. The value of $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{b}(x^2+y^2+z^2)dxdydz$ is

(a) $\left(8abc(a^2+b^2+c^2)\right)$

(b) $\left(\frac{4}{3}abc(a^2+b^2+c^2)\right)$

(c) $\left(\frac{8}{3}abc(a^2+b^2+c^2)\right)$

(d) None of these

8. The value of $\int_{0}^{\ln 2}\int_{0}^{x}\int_{0}^{x+\ln y} e^{x+y+z}dzdydx$ is

(a) $\left(\frac{8}{3}\ln 2-\frac{19}{9}\right)$ (b) $\left(\frac{8}{3}\ln 2+\frac{19}{9}\right)$

(c) $\left(\frac{4}{3}\ln 2-\frac{19}{9}\right)$ (d) None of these

9. The value of $\int_{0}^{1}\int_{y^2}^{1}\int_{0}^{1-x} xdzdxdy$ is

(a) $\left(\frac{-4}{35}\right)$ (b) $\left(\frac{2}{35}\right)$

(c) $\left(\frac{4}{35}\right)$ (d) None of these

10. The value of $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y}\frac{dzdydx}{(1+x+y+z)^3}$ is

(a) $\left(\frac{-1}{2}\left(\ln 2-\frac{5}{8}\right)\right)$ (b) $\left(\frac{1}{2}\left(\ln 2-\frac{5}{8}\right)\right)$

(c) $\left(\frac{1}{2}\left(\ln 2+\frac{3}{8}\right)\right)$ (d) None of these

11. The value of $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xyzdzdydx$ is

(a) $\left(\frac{1}{48}\right)$ (b) $\left(\frac{1}{24}\right)$

(c) $\left(\frac{1}{12}\right)$ (d) None of these

12. The value of $\int_{1}^{e}\int_{1}^{\ln y}\int_{1}^{e^x}\log zdzdxdy$ is

(a) $\left(\frac{e^2}{4}+2e+\frac{13}{4}\right)$ (b) $\left(-\frac{e^2}{4}+2e-\frac{13}{4}\right)$

(c) $\left(\frac{e^2}{4}+2e-\frac{13}{4}\right)$ (d) $\left(\frac{e^2}{4}-2e+\frac{13}{4}\right)$

13. The value of $\int_1^3\int_{1/x}^1\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx$ is

(a) $\left(\frac{1}{6}\left[\frac{13}{3}+\ln 3\right]\right)$ (b) $\left(\frac{1}{6}\left[\frac{-26}{3}+\ln 3\right]\right)$

(c) $\left(\frac{2}{6}\left[\frac{26}{3}+\ln 3\right]\right)$ (d) $\left(\frac{1}{6}\left[\frac{26}{3}-\ln 3\right]\right)$

14. The value of $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$ is

(a) $\left(\frac{5a^3\pi}{64}\right)$ (b) $\left(\frac{5a^3\pi}{32}\right)$

(c) $\left(\frac{5a^3\pi}{16}\right)$ (d) $\left(\frac{5a^3\pi}{8}\right)$

15. The value of $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$ is

(a) $\left(\frac{1}{4}(e^{4a}-6e^{2a}+8e^a-3\right)$

(b) $\left(\frac{-1}{8}(e^{4a}-6e^{2a}+8e^a-3\right)$

(c) $\left(\frac{1}{8}(e^{4a}-6e^{2a}+8e^a-3\right)$

(d) None of these

16. The value of $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$ is

(a) 32π (b) 8π (c) 16π (d) 4π

17. The value of $\iiint (z^5+z)\,dz\,dx\,dy$ over the sphere $x^2+y^2+z^2=1$ is

(a)1 (b) 0 (c) 3 (d) 4

18. Find the volume bounded by the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}=1$.

(a) $\frac{2\pi abc}{5}$ (b) $\frac{4\pi abc}{5}$

(c) $\frac{16\pi abc}{5}$ (d) $\frac{8\pi abc}{5}$

19. The value of $\iiint_V z\,dV$, where V is the volume bounded below by the cone $x^2+y^2=z^2$ and above by the sphere $x^2+y^2+z^2=1$, lying on the positive side of the y-axis.

(a) $\frac{\pi}{16}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{32}$

20. Find the volume cut-off the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$.

(a) $\frac{\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$ (b) $\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$

(c) $\frac{3\pi}{3}\left(1+\frac{1}{\sqrt{2}}\right)$ (d) None of these

21. The integral $\int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx$ is equal to

(a) $\int_0^1\int_0^x\int_z^x f(x,y,z)\,dy\,dz\,dx$

(b) $\int_0^1\int_0^x\int_0^z f(x,y,z)\,dy\,dz\,dx$

(c) $\int_0^1\int_x^1\int_z^x f(x,y,z)\,dy\,dz\,dx$

(d) $\int_0^1\int_x^1\int_0^z f(x,y,z)\,dy\,dz\,dx$

22. Volume of the part of the cylinder $\frac{x^2}{1}+\frac{z^2}{4}=1$, which lies between the planes $y=0$ and $y=3x$.

(a) 8 (b) 6 (c) 4 (d) 2

23. Find the volume of the portion cut off from the cylinder $2x^2+y^2=4x$ and the plane $z=x$ and $z=3x$

(a) $2\sqrt{2}\,\pi$ (b) $\sqrt{2}\,\pi$ (c) $\sqrt{3}\,\pi$ (d) $3\sqrt{3}\,\pi$

24. Find the volume generated by revolving the region bounded by $y=\sqrt{x}$; $y=2$; $x=0$ about the x-axis

(a) 2π (b) 4π (c) 8π (d) 16π

25. Consider the solid obtained by revolving the region bounded by the surface $y = x^2 + x + 1$; $y = 1$ and $x = 1$ about the line x = 2 then volume generated is

(a) $\frac{26\pi}{6}$ (b) $\frac{13\pi}{6}$ (c) $\frac{13\pi}{12}$ (d) $\frac{13\pi}{24}$

26. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find volume of solid

(a) $\frac{117\pi}{6}$ (b) $\frac{117\pi}{3}$ (c) 7π (d) $\frac{7\pi}{3}$

27. Find the integral $\iiint_R \sqrt{1 - x^2 - y^2 - z^2}\, dx\, dy\, dz$ where R is region interior to the sphere $x^2 + y^2 + z^2 = 1$.

(a) $\frac{\pi^2}{2}$ (b) 1 (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{4}$

28. Find the volume of the solid which is bounded by paraboloids $z = x^2 + y^2$ and $z = 4 - 3(x^2 + y^2)$.

(a) -2π (b) $\pi/2$ (c) 2π (d) $-\pi/2$

29. Through a diameter of the upper base of a sight cylinder of altitude 'a' and radius 'r' pass two planes which touch the lower base on opposite side. Find the volume of the cylinder included between the 2 planes.

(a) $ar^2\left[\pi + \frac{4}{3}\right]$ (b) $ar^2\left[\pi - \frac{4}{3}\right]$

(c) $\pi - \frac{4}{3}$ (d) $\pi + \frac{4}{3}$

30. Volume bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and cone $x^2 + y^2 = z^2$ above xy-plane is equal to

(a) $\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$ (b) $\frac{2\pi}{3}\left(1 + \frac{1}{\sqrt{2}}\right)$

(c) $\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{3}}\right)$ (d) none

31. Volume generated by revolving parabola $y^2 = 4ax$ about the latus rectum

(a) $\frac{16}{15}\pi a^3$ (b) $\frac{15}{16}\pi a^3$ (c) $\frac{1}{15}\pi a^3$ (d) none

32. The volume of the solid bounded by cylinder $x^2 + y^2 = 1$ and paraboloid $z = x^2 + y^2$ where $x > 0, y < 0, z > 0$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{8}$ (d) none

33. Find the volume of the solid which is below the plane $z = 2x + 3$ and above the xy-plane and bounded by $y^2 = x$, $x = 0$ and $x = 2$.

(a) $\frac{14\sqrt{2}}{5}$ (b) $\frac{7\sqrt{2}}{5}$

(c) $14\sqrt{2}$ (d) None of these

34. Find the volume of the solid which is below the plane $z = x + 3y$ and above the ellipse $25x^2 + 16y^2 = 400$, $x \geq 0, y \geq 0$.

(a) $\frac{382}{3}$ (b) $\frac{380}{3}$ (c) $\frac{119}{3}$ (d) $\frac{190}{3}$

35. Find the volume of the solid which is bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z = 1$ and $z = 0$.

(a) 2π (b) 3π (c) π (d) 4π

PART-B (Multiple Select Questions (MSQ))

1. Let $I = \iiint_R (2x + y)\, dx\, dy\, dz$ where R is the closed region bounded by the cylinder $z = 4 - x^2$ and the plane $x = 0$, $y = 0$, $y = 2$ and $z = 0$. Let $J = \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z - x^2}} dy\, dx\, dz$ then

(a) $I = \frac{80}{3}$ (b) $I = \frac{40}{3}$

(c) $J = 8\pi$ (d) $J = 4\pi$

2. Which of the following is TRUE ?

(a) Volume bounded by $z = 2xy$; $z = 0$; $y = 1$; $y = 2$; $x = 0$; $x = 1$ is 1/2

(b) Volume bounded by $z = 2xy$; $z = 0$; $y = 1$; $y = 2$; $x = 0$; $x = 1$ is 3/2

(c) Volume bounded by paraboloid $x^2 + y^2 = 1 + z$ and $z = 0$ is $\pi/2$

(d) Volume bounded by paraboloid $x^2 + y^2 = 1 + z$ and $z = 0$ is $(-\pi/2)$

PART-C (Numerical Answer Type (NAT))

1. The value of $\iiint_v (x+y+z)dv$ whre v is cube $x>0, y>0, z\geq 0, x\leq 1, y\leq 1$ and $z\leq 1$ is

2. The value of $\int_0^a\int_0^x\int_0^y xyzdzdydx$ is

3. $\iiint_V \frac{dxdydz}{(x+y+z+1)^3}$, V is the domain bounded by the planes $x=0, y=0, z=0, x+y+z=1$ is

4. The value of $\iiint_V xydxdydz$, v is the domain bounded by the hyperbolic paraboloiad $z = xy$ and planes $x + y = 1$ and $z = 0$ $(z\geq 0)$ is

5. $\iiint_v y\cos(x+z)dxdydz$, v is the domain bounded by the cylinder $y=\sqrt{x}$ and the planes $y = 0, z = 0$ and $x+z=\frac{\pi}{2}$

6. The volume of the solid bounded by $z = 4 - y^2$ and $z = y^2 + 2$ and planes $x = -1$ and $x = 2$ is

7. The volume of the solid bounded by the paraboloids $z=x^2+y^2$ and $z=x^2+2y^2$ and the planes $y = x, y = 2x, x = 1$ is

8. The value of $\int_0^2\int_0^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2dzdxdy$ is

9. The value of

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dzdxdy \text{ is}$$

10. The value of $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{a^2-x^2-y^2} x^2dzdydx, (a>0)$ is

11. The value of $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}}dzdydx$ is

12. The value of $\iiint_G xyzdV$, where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0, y = x$ and $y = 0$

13. The volume of the solid in the first octant bounded by the co-ordinate planes and the plane $3x+6y+4z=12$ is

14. The volume the solid bounded by the surface $z=\sqrt{y}$ and the planes $y+z=4$ and $z=0$ is

15. The value of $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zrdzdrd\theta$ is

16. The value of $\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r\sin\theta dzdrd\theta$ is

17. The value of $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^3\sin\phi\cos\phi drd\phi d\theta$ is

18. The volume of the solid bounded by the parabola $z = x^2 + y^2$ and $z=2x^2+2y^2$, cylinder $y = x^2$ and plane $y = x$ is

19. The volume of the solid bounded by $z=x^2+y^2$ and plane $z=x+y$.

20. The value of $\int_0^2\int_{-1}^{y^2}\int_{-1}^{z} yzdxdydz$ is

21. The value of $\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^x xydxdydz$ is

22. The value of $\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} zdzdydx$ is

23. The value of $\iiint_G xy\sin(yz)dv$, where G is the rectangular box defined by the inequalities $0\leq x\leq \pi, 0\leq y\leq 1, 0\leq z\leq \pi/6$

24. $\iiint_W (x^2+y^2+2z^2)dv$, where w is the region that lies below the paraboliod $z = 25 - x^2 - y^2$, is the cylinder $x^2 + y^2 = 4$ and above xy-plane is

25. The volume of the solid that lies behind the plane $x + y + z = 8$ and in front of the region in the yz-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$ is

26. $\iiint_E \sqrt{3x^2+3z^2}\,dv$, where E is the solid bounded by $y=2x^2+2z^2$ and the plane $y = 8$ is

27. Evaluate $\iiint_E 12y-8x\,dV$ where E is the region behind $y = 10 - 2z$ and in front of the region in the xy-plane bounded by $z = 2x$, $z = 5$ and $x = 0$.

28. Evaluate $\iiint_E yz\,dV$ where E is the region bounded by $x = 2y^2 + 2z^2 - 5$ and the plane $x = 1$

29. Evaluate $\iiint_E 15z\,dV$ where E is the region between $2x + y + z = 4$ and $4x + 4y + 2z = 20$ that is in front of the region in the xy-plane bounded by $z = 2y^2$ and $z = \sqrt{4y}$

30. Evaluate $\int_0^1\int_0^{z^2}\int_0^3 y\cos(z^5)\,dx\,dy\,dz$

31. Evaluate $\iiint_E 6z^2\,dV$ where E is the region below $4x + y + 2z = 10$ in the first octant.

32. Evaluate $\iiint_E 3-4x\,dV$ where E is the region below z = 4 – xy and above the region in the xy-plane defined by $0 \le x \le 2, 0 \le y \le 1$

33. Use a triple integral to determine the volume of the region below z = 4 – xy and above the region in the xy-plane defined by $0 \le x \le 2, 0 \le y \le 1$.

34. Use a triple integral to determine the volume of the region that is below $z=8-x^2-y^2$ above $z=-\sqrt{4x^2+4y^2}$ and inside $x^2+y^2=4$.

35. The loop of the curve $2y^2 = x(x-1)^2$ revolves about the line $y = 1$, the volume of solid generated is $\frac{8\sqrt{2}K}{15}$, what is K______.

36. Volume of solid whose base is in the xy-plane and is the triangle bounded by the x-axis, the line $y = x$ and the line $x = 1$; while top of the solid is in the plane $z = x + y + 1$______.

37. Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ throughout the volume bounded by the co-ordinate planes and the plane $x+y+z=1$ is $K\log 2 - L\frac{10}{16}$ then $K-L$ equals______.

38. Find the volume cut from a sphere of radius 5(with centre origin) by a right circular cylinder with 3 as a radius of the base and whose axes passes through the centre of the sphere is $K\,62\pi$ then K is______.

39. Find the volume of cylindrical column standing on the area common to the parabola $x = y^2$; $y = x^2$ as base and cut off by the surface $z = 1 + y - x^2$ ______.
(Correct upto three decimal places)

40. Find the volume of the solid cut off by the surface $z = (x+y)^2$ from the right prism whose base in the plane $z = 0$ is the rectangle bounded by the line $x = 0$; $y = 0$; $x + y = 1$______.

41. Volume of the solid bounded by the surface $(x^2+y^2+z^2)^3 = 27xyz$ ______.

42. Volume common to $x^2+y^2=1$, $y^2+z^2=1$, and $x^2+z^2=2$ has value $k(2-\sqrt{2})$ then find k____

43. Find the volume of the solid which is bounded by the paraboloid $z = 9 - x^2 - 4y^2$ and the coordinate planes $x \ge 0,\ y \ge 0,\ z \ge 0$.

44. Find the volume of the solid which is bounded by the surfaces $2z = x^2 + y^2$ and $z = x$.

45. Find the volume of the solid which is bounded by the surfaces $z = 0,\ 3z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 9$.

46. Find the volume of the solid which is in the first octant bounded by the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

47. Find the volume of the solid which is bounded by the paraboloid $4z = x^2 + y^2$, the cone

$z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 2x$.

48. Find the volume of the solid which is common to the right circular cylinders $x^2 + z^2 = 1,\ y^2 + z^2 = 1$ and $x^2 + y^2 = 2x$.

49. Find the volume of the solid which is above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + (z-a)^2 = a^2$.

50. Find the volume of the solid which is below the surface $z = 4x^2 + 9y^2$ and above the square with vertices at (0, 0), (2, 0), (2, 2) and (0, 2).

51. Find the volume of the solid which is bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - 3(x^2 + y^2)$.

52. Find the volume bounded by $y^2 = x+1,\ y^2 = -x+1,\ z = -2,\ z = x+4$.

53. The volume bounded by the paraboloid $y^2 + z^2 = 4ax$ and cylinder $x^2 + y^2 = 2ax$ is

54. Find the volume enclosed by the surfaces, $x^2 + y^2 = cz,\ x^2 + y^2 = ax,\ z = 0$.

PART-A (Multiple Choice Questions (MCQ))

1. (a)	**2. (a)**	**3. (b)**	**4. (c)**	**5. (c)**	**6. (b)**
7. (c)	**8. (a)**	**9. (c)**	**10. (b)**	**11. (a)**	**12. (d)**
13. (d)	**14. (a)**	**15. (c)**	**16. (b)**	**17. (b)**	**18. (d)**
19. (a)	**20. (b)**	**21. (a)**	**22. (a)**	**23. (a)**	**24. (c)**
25. (b)	**26. (b)**	**27. (d)**	**28. (c)**	**29. (b)**	**30. (b)**
31. (a)	**32. (c)**	**33. (a)**	**34. (b)**	**35. (c)**	

PART-B (Multiple Select Questions (MSQ))

1. (a,c) **2. (b,c)**

PART-C (Numerical Answer Type (NAT))

1. $\left(\frac{3}{2}\right)$	**2.** $\left(\frac{a^6}{48}\right)$	**3.** $\left(\frac{1}{2}\left(\ln 2 - \frac{5}{8}\right)\right)$	**4.** $\left(\frac{1}{180}\right)$
5. $\left(\frac{\pi^2}{16} - \frac{1}{2}\right)$	**6. (8)**	**7.** $\left(\frac{7}{12}\right)$	**8.** $\left(\frac{32(2\sqrt{2}-1)\pi}{15}\right)$
9.	**10.** $\left(\frac{\pi a^6}{48}\right)$	**11.**	**12.** $\left(\frac{1}{6}\right)$
13. (4)	**14.** $\left(\frac{256}{15}\right)$	**15.** $\left(\frac{\pi}{4}\right)$	**16.** $\left(\frac{\pi}{16}\right)$
18 $\left(\frac{3}{35}\right)$	**19.** $\left(\frac{\pi}{2}\right)$	**20.** $\left(\frac{47}{3}\right)$	**21.** $\left(\frac{81}{5}\right)$

22. $\left(\frac{128}{5}\right)$	**23.** $\pi(\pi-3)/2$	**24.** **(2300 π)**	**25.** $\left(\frac{49}{5}\right)$
26. $\left(\frac{256\sqrt{3}\pi}{15}\right)$	**27.** $\frac{3125}{6}$	**28.** **0**	**29.** $\frac{49}{2}$
30. $\left(\frac{3}{10}\sin 1\right)$	**31.** $\left(\frac{625}{2}\right)$	**32.** $-\left(-\frac{17}{3}\right)$	**33.** **(7)**
34. $\frac{104\pi}{3}$	**35.** **3.14**	**36.** **1**	**37.** **0**
38. **0.67**	**39.** **0.596**	**40.** **0.25**	**41.** **4.5**
42. **8**	**43.** $\frac{81\pi}{16}$	**44.** $\frac{\pi}{4}$	**45.** $\frac{27\pi}{2}$
46. $\frac{2a^3}{3}$	**47.** $\frac{(256-27\pi)}{72}$	**48.** $8\left(2-\sqrt{2}\right)$	**49.** πa^3
50. $\frac{208}{3}$	**51.** 2π	**52.** $\frac{48}{3}$	**53.** $\frac{16a^3}{3+2\pi a^3}$.
54. $\frac{3\pi a^2}{32c}$			