CHAPTER - 2

UNITS, MEASUREMENTS, DIMENSIONAL AND ERROR ANALYSIS

PHYSICAL QUANTITIES:

To know and understand a physical situation completely, one has to measure the quantities like mass, distance, time, acceleration, force etc. All these quantities are known as physical quantities. In physics, we study about physical quantities and their interrelationships.

MEASUREMENT:

For the measurement of a physical quantity we need some standard unit of that quantity. Thus, measurement is the comparison of a quantity with a standard of the same physical quantity.

UNITS:

A chosen standard of some kind taken as reference for the measurement of physical quantity is called the unit of that physical quantity.

Example: meter, second, kilogram etc.

The basic properties of units are:

- They must be well defined.
- They should be easily available and reproducible.
- They should be invariable. For example, 1 step as a unit of length is variable.
- They should be accepted to all.

FUNDAMENTAL QUANTITIES AND FUNDAMENTAL UNITS:

Physical quantities which do not depend on other quantities are known as fundamental quantities, and their units are called fundamental units.

S.I. SYSTEM:

In this system, there are seven fundamental quantities which are shown below.

Physical Quantity	Name	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Temperature	kelvin	k
Luminous Intensity	candela	cd
Electric Current	ampere	A
Amount of Substance	mole	mol

There are two supplementary units in this system which are as follows:

Physical Quantity	Name	Symbol
Phase angle	radian	rad
Solid angle	Steradian	Sr

DERIVED QUANTITIES AND DERIVED UNITS:

The units of the physical quantities which depend upon fundamental units or they can be derived from fundamental units are known as derived units and quantities are known as derived quantities.

Some S.I. Derived Units:

Derived Quantity	Derived Units	In terms of Fundamental Units
Force	newton	$kg\ ms^{-2}$
Resistance	ohm	$kg\ m^2s^{-3}A^{-2}$
Magnetic field	tesla	$kg\ s^{-2}\ A^{-1}$
Electric potential	volt	$kg\ m^2s^{-3}A^{-1}$
Power	watt	$kg\ m^2s^{-3}$
Electric charge	coulomb	As
Capacitance	farad	$kg^{-1}\ m^{-2}s^4\ A^2$
Work or energy	joule	$kg\ m^2s^{-2}$
Inductance	henry	$kg\ m^2s^{-2}\ A^{-2}$
Frequency	hertz	s^{-1}

DIMENSIONS OF A PHYSICAL QUANTITY:

The dimensions of a physical quantity may be defined as the powers to which the fundamental quantities must be raised to represent the derived unit of physical quantity.

For example: Let us consider the derived unit of area.

Area: Length × Length = $[L] \times [L] = [L]^2$

To represent area, we have to raise [L] to the power two. So, the dimension of area is L^2.

Another example: to represent density

$$\text{Density} = \frac{\text{mass}}{\text{volume}} = \frac{\text{mass}}{(\text{length})^3}$$

$$= (\text{mass}) \times (\text{length})^{-3} = [M]\ [L^{-3}] = [M^1L^{-3}T^0]$$

So, the dimension of density are: +1 in mass, –3 in length and zero in time.

Dimensional Formulae and Dimensional Equations:

Dimensional Formula: It is defined as an expression showing how and which of the fundamental units enter into the derived unit of a physical quantity.

The dimensional formula of density is $[M^1L^{-3}T^0]$

Dimensional Equation: It is defined as an equation obtained by equating the physical quantity with its dimensional formula.

For example, the dimensional formula for velocity is $[LT^{-1}]$, when it is equated with velocity, then

$$[V] = [LT^{-1}]$$

This is known as dimensional equation. In general, the dimensional equation is expressed as

$$[X] = [M^a L^b T^c]$$

Some physical quantities and their dimensional formulae:

Physical Quantity	Dimensional Formula
Velocity = length/time	$[LT^{-1}]$
Acceleration = velocity/time	$[LT^{-2}]$
Momentum = mass × velocity	$[MLT^{-1}]$
Force = mass × acceleration	$[MLT^{-2}]$
Work = force × distance	$[ML^2T^{-2}]$
Power = work/time	$[ML^2T^{-3}]$
Pressure = force/area	$[ML^{-1}T^{-2}]$
Stress = force/area	$[ML^{-1}T^{-2}]$
Strain = (ratio)	dimensionless
Coefficient of elasticity = stress/strain	$[ML^{-1}T^{-2}]$
Torque = force × distance	$[ML^2T^{-2}]$
Angular velocity = angle/time	$[L^0T^{-1}]$
Moment of inertia = mass × $(\text{distance})^2$	$[ML^2]$
Angular momentum = moment of inertia, angular velocity	$[ML^2T^{-1}]$
Latent heat = heat/mass	$[L^2T^{-2}]$
Thermal resistance = $\frac{\text{temperature} \times \text{time}}{\text{heat}}$	$[M^{-1}L^{-2}T^3\theta]$

List of physical quantities which have same dimensional formula:

Quantities	Dimensional Formula
Frequency, angular velocity, angular frequency, velocity gradient	$[M^0L^0T^{-1}]$
Speed, velocity, distance covered in n^{th} second	$[M^0LT^{-1}]$
Acceleration, retardation, gravitational intensity	$[M^0LT^{-2}]$
Force, weight, thrust, energy gradient	$[MLT^{-2}]$
Work, internal energy, torque, quantity of heat, quantity of light	$[ML^2T^{-2}]$
Pressure, stress, modulus of elasticity (Young's modulus, bulk modulus, rigidity modulus)	$[ML^{-1}T^{-2}]$
Surface tension, surface energy, force gradient, spring constant	$[ML^0T^{-2}]$
Angular momentum, Planck's constant	$[ML^2T^{-1}]$
Thermal capacity, entropy, Boltzmann's constant	$[ML^2T^{-2}\theta^{-1}]$
Electric potential, electric potential difference, electromotive force	$[ML^2T^{-3}A^{-1}]$
Electric field strength, electric potential gradient	$[MLT^{-3}A^{-1}]$

Principle of Homogeneity of Dimensions:

If an equation truly expresses a proper relationship among variables in a physical process, then it will be dimensionally homogeneous. The equations are correct for any system of units and consequently each group of terms in the equation must have the same dimensional representation. This is also known as the principle of homogeneity of dimensions. The dimensional homogeneity imposes the following constraints on any mathematical representation of a relationship:

(1) Both sides of the equation must have the same dimension

(2) Wherever a sum of quantities appears in a function, all the terms in the sum must have the same dimension.

(3) All arguments of any exponential, logarithmic, or other special functions that appear in the given function must be dimensionless.

For example, if a physical equation is represented by $A = Be^{-C} - \frac{(D_1 + D_2)}{E} + F$

then, C must be dimensionless, D_1 and D_2 must have the same dimension, and A, B, D/E and F must have the same dimension.

An important consequence of dimensional homogeneity is that the form of a physical equation is independent of the size of the base units.

Applications of Dimensional Analysis:

(1) To convert units of a physical quantity from one system of units to another:

It is based on the fact that - Numerical value (n) × unit (u) = constant

So, on changing unit, numerical value will also change. If n_1 and n_2 are the numerical values of a given physical quantity and u_1 and u_2 be the units respectively in two different system of units, then $n_1u_1 = n_2u_2$.

$$\therefore\ n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

Example-1: The value of $G = 6.67 \times 10^{-11}$ N-m^2/kg^2 in S.I. units. Convert it into C.G.S. system of units.

Soln. The dimensional formula for gravitational constant G is $[M^{-1}L^3T^{-2}]$. Let its value in C.G.S. system be n_2.

Then, $n_1[M_1^{-1}L_1^3T_1^{-2}] = n_2[M_2^{-1}L_2^3T_2^{-2}]$

or, $n_2 = n_1\left[\frac{M_1}{M_2}\right]^{-1}\left[\frac{L_1}{L_2}\right]^3\left[\frac{T_1}{T_2}\right]^{-2}$

Here, M_1 = 1 kg, M_2 = 1g = 10^{-3} kg, L_1 = 1m, L_2 = 1cm = 10^{-2} m, $T_1 = T_2$ = 1 s

$$\therefore\ n_2 = 6.67\times10^{-11}\left[\frac{1\text{kg}}{10^{-3}\text{kg}}\right]^{-1}\left[\frac{\text{m}}{10^{-2}\text{m}}\right]^3\left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

$= 6.67 \times 10^{-8}$ dyne cm^2/g^2.

(2) To check the dimensional correctness of a given physical relation: It is based on principle of homogeneity, which states that a given physical relation is dimensionally correct if the dimensions of the various terms on either side of the relation are the same.

Example-2: Check the correctness of the formula $T^2 = \frac{4\pi^2 a^3}{GM}$, where '$T$' represents the time period, 'G' gravitational constant, 'M' mass and 'a' the radius of the orbit.

Soln. Using the principle of homogeneity, we have

$$\text{R.H.S.} = \left[\frac{L^3}{(M^{-1}L^3T^{-2})(M)}\right] = [T^2] = \text{L.H.S.}$$

Therefore, the formula is correct.

(3) To establish a relation between different physical quantities: The principle of homogeneity also enables us to deduce a relationship between different physical quantities.

Example-3: Deduce an expression for the time period of a simple pendulum.

Soln. The time period of a simple pendulum depends upon the following factors:

(i) mass of the bob (m)
(ii) length of simple pendulum (l)
(iii) acceleration due to gravity (g) and
(iv) angle of swing of simple pendulum (θ)

Let $t = k\, m^a l^b g^c \theta^d$, where k is constant of proportionality. Taking dimensions on both sides, we have,

$[T] = [M]^a[L]^b[LT^{-2}]^c$

or, $T = M^a L^{b+c} T^{-2c}$

Equating powers of M, L and T on both sides $a = 0, b + c = 0$ and $-2c = 1$

$\therefore\ c = -\frac{1}{2}, b = \frac{1}{2}$ and $a = 0$

Thus, $t = k\, l^{1/2} g^{-1/2}$ or $t = k\sqrt{\frac{l}{g}}$

Experimentally it is observed that $k = 2\pi$

$\therefore t = 2\pi\sqrt{\frac{l}{g}}$

SIGNIFICANT FIGURES AND ORDER OF ACCURACY

The accuracy of a number is specified by the number of significant figures it contains. A significant figure is any digit, including zero, provided it is not used to specify the location of the decimal point for the number.

For example, the numbers begin or end with zeros, however, it is difficult to tell how many significant figures are in the number. Consider the number 400. Does it have one (4) or perhaps two (40) or three (400) significant figures? In order to clarify this situation, the number should be reported using powers of 10.

Using engineering notation, the exponent is displayed in multiples of three in order to facilitate conversion of SI units to those having an appropriate prefix. Thus, 400 expressed to one significant figure would be $0.4(10^3)$. Likewise, 2500 and 0.00546 expressed to three significant figures would be $2.50(10^3)$ and $5.46(10^{-3})$.

Rules to Determine Significant Figures:

- Zero is not a significant figure when it is the first figure in a number (e.g. 0.00034 has only two significant figures). A zero in any other position is significant (e.g. 102 has three significant figures). In order to avoid confusion it is preferable to use scientific notation when expressing results (e.g. 6.20×10^4 has three significant figures).
- When rounding off numbers, add one to the last figure retained if the following figure is greater than 5 (e.g. 0.53257 becomes 0.5326 when rounded off to four significant figures).
- Round 5 to the nearest even number (e.g. 0.255 becomes 0.26 when rounded off to two significant figure). If the digit just before 5 is even, it is left unchanged (e.g. 0.345 becomes 0.34 when rounded off to two significant figures); if it is odd, its value is increased by one (e.g. 0.335 becomes 0.34 when rounded off to two significant figures).
- If two or more figures are present to the right of the figure to be retained, they are considered as a group (e.g. 6.8[501] should be rounded off to 6.9; 7.4[499] should be rounded off to 7.4)
- In addition and subtraction the result should be reported to the same number of decimal places as there are in the number with the smallest number of decimal places.
- In multiplication and division the result should have an uncertainty of the same order as the number with the greatest uncertainty.
- In the logarithm of a number we retain the same number of digits to the right of the decimal point as there are significant figures in the original number.
- In the antilogarithm of a number we retain as many significant figures as there are digits to the right of the decimal point in the original number.

ERRORS OF MEASUREMENT

The measuring process is essentially a process of comparison. In spite of our best efforts, the measured value of a quantity is always somewhat different from its actual value, or true value. This difference in the true value and measured value of a quantity is called error of measurement.

1. **Absolute Error:** Absolute error in the measurement of a physical quantity is the magnitude of the difference between the true value and the measured value of the quantity.

   Let a physical quantity be measured n times. Let the measured value be $a_1, a_2, a_3, .., a_n$. The arithmetic mean of these value is $a_m = \frac{a_1 + a_2 + ... + a_n}{n}$

   Usually, a_m is taken as the true value of the quantity, if the same is unknown otherwise.

   By definition, absolute errors in the measured values of the quantity are

   $\Delta a_1 = a_n - a_1, \Delta a_2 = a_m - a_2$, ..., $\Delta a_2 = a_m - a_n$

   The absolute errors may be positive in certain cases and negative in certain other cases.

2. **Mean Absolute Error:** It is the arithmetic mean of the magnitudes of absolute errors in all the measurements of the quantity. It is represented by $\overline{\Delta a}$. Thus,

   $$\overline{\Delta a} = \frac{|\Delta a_1| + |\Delta a_2| + ... + |\Delta a_n|}{n}$$

   Hence the final result of measurement may be written as $a = a_m \pm \overline{\Delta a}$. This implies that any measurement of the quantity is likely to lie between $(a_m + \overline{\Delta a})$ and $(a_m - \overline{\Delta a})$.

3. **Relative Error or Fractional Error:** The relative error or fractional error of measurement is defined as the ratio of mean absolute error to the mean value of the quantity measured. Thus

   Relative error or Fractional error $= \frac{\text{Mean absolute error}}{\text{Mean value}} = \frac{\overline{\Delta a}}{a_m}$

4. **Percentage Error:** When the relative/fractional error is expressed in percentage, we call it percentage error.

   Thus, percentage error $= \frac{\overline{\Delta a}}{a_m} \times 100$

Propagation of Errors

In the following, Δa = absolute error in measurement of a, Δb = absolute error in measurement of b, Δx = absolute error in calculation of x.

1. **Error in sum of the quantities :** Suppose $x = a + b$

   The maximum absolute error in x is $\Delta x = \pm(\Delta a + \Delta b)$

   Percentage error in the value of $x = \frac{(\Delta a + \Delta b)}{a + b} \times 100$

2. **Error in difference of the quantities :** Suppose $x = a - b$

   The maximum absolute error in x is $\Delta x = \pm(\Delta a + \Delta b)$

   Percentage error in the value of $x = \frac{(\Delta a + \Delta b)}{a - b} \times 100$

3. Error in product of quantities : Suppose $x = a \times b$

The maximum fractional error in x is $\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$

Percentage error in the value of x = (% error in value of a) + (% error in value of b)

4. Error in division of quantities : Suppose $x = ab$

The maximum fractional error in x is $\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$

Percentage error in the value of x = (% error in value of a) + (% error in value of b)

5. Error in quantity raised to some power : Suppose $x = \frac{a^n}{b^m}$

The maximum fractional error in x is $\frac{\Delta x}{x} = \pm\left(n\frac{\Delta a}{a} + m\frac{\Delta b}{b}\right)$

Percentage error in the value of $x = n$ (% error in value of a) + m (%error in value of b)

SOLVED EXAMPLES

1. One Angstrom (Å) is **[JNU Life Sc. 2004]**

(a) 10^{-10} meters (b) 10 cm (c) 10^{-5} meters (d) 10^{-2} cm

Soln. One Angstrom is equal to 10^{-10} m.

Correct option is (a)

2. The dimension of energy is **[JNU Life Sc. 2011]**

(a) $ML^{-1}T^{-2}$ (b) ML^2T^{-2} (c) $M^2L^2T^{-2}$ (d) MLT^{-2}

Soln. One of the ways of deriving the dimension of energy is using the formula for work done.

Work done, $W = \vec{F}.\vec{S} = FS\cos\theta$

$[W] = [M][LT^{-2}][L] = [MLT^{-2}L] = [ML^2T^{-2}]$

Therefore, the dimension of energy is ML^2T^{-2}

Correct option is (b)

3. The dimensional representation of Planck's constant is same as that of: **[TIFR 2014]**

(a) Angular momentum (b) Momentum

(c) Torque (d) Energy

Soln. The dimensional representation of Planck's constant: ML^2T^{-1}

Dimensional formulae for the following quantities:

(a) Angular momentum : ML^2T^{-1}

(b) Momentum : MLT^{-1}

(c) Torque : ML^2T^{-2}

(d) Energy : ML^2T^{-2}

Correct option is (a)

4. Which of the following correctly expresses the Planck length (I) in terms of other fundamental constants (the gravitational constant G, velocity of light c and Planck's constant $\hbar$) ? **[TIFR 2015]**

(A) $\sqrt{\frac{\hbar G}{c^3}}$ (B) $\sqrt{\frac{c^3}{\hbar G}}$ (C) $\sqrt{\frac{c}{\hbar^2 G}}$ (D) $\sqrt{\frac{\hbar G^2}{c^2}}$

(a) A (b) B (c) C (d) D

Soln. Dimension of Planck's length l : M^0LT^0

Dimension of reduced Planck's constant $\hbar$: ML^2T^{-1}
Dimension of gravitational constant G : $M^{-1}L^3T^{-2}$
Dimension of speed of light c : LT^{-1}

Let $l = \hbar^a G^b c^x$
$[L] = [ML^2T^{-1}]^a [M^{-1}L^3T^{-2}]^b [LT^{-1}]^x = [M^{a-b} L^{2a+3b+x} T^{-a-2b-x}]$
Comparing the powers on both sides,
$a - b = 0$ ...(i)
$2a + 3b + x = 1$ ...(ii)
$-a - 2b - x = 0$ ...(iii)
$\Rightarrow a = b$
Using (i) in (ii), $2b + 3b + x = 1 \Rightarrow 5b + x + 1$ ...(iv)
Using (i) in (iii), $-b - 2b - x = 0 \Rightarrow x = -3b$ ...(v)

Using (i) in (iv), $5b - 3b = 1 \Rightarrow b = \frac{1}{2}$

Planck's length $l = h^{1/2}G^{1/2}c^{-3/2} = \sqrt{\frac{hG}{c^3}}$

Correct option is (a)

5. The drag force on a particle moving at a speed v in a medium is of the form $F = \zeta v$, where the drag coefficient ζ depends on the particle's shape and size and on properties of the medium. If length is measured in cm, mass in g, and time in s, then ζ has units: **[TIFR 2018]**

(a) g (b) g/(cm s)
(c) g/s (d) g cm^2/s

Soln. Given: $F = \zeta v$

Units of $\zeta \equiv \frac{[F]}{[v]} = \frac{\text{g cm s}^{-2}}{\text{cm s}^{-1}} = \text{gs}^{-1}$

Correct option is (c)

6. A tachometer is device to measure **(JNU Biotech, 2003)**

(a) gravitational pull (b) speed of rotation (c) tension of a spring (d) surface tension

Soln. A tachometer is a device to measure speed of rotation.

Correct option is (c)

7. Which one of the following do not have the same dimension? **(JNU Biotech, 2006)**

(a) Planck's constant and energy (b) Work and energy
(c) Angle and strain (d) Relative density and refractive index

Soln. Work and energy have dimension $[ML^2T^{-2}]$

Angle, strain, relative density and refractive index are dimensionless quantities.

Planck's constant h : $[ML^2T^{-1}]$

Therefore, the pair Planck's constant and energy, do not have same dimension.

Correct option is (a)

8. In arithmetic $17.8 \times 3.1143 = 55.4354$. But as a result or experimental measurements the best way to express the product is **(JNU Biotech, 2007)**

(a) 55.4354 (b) 55.4 (c) 55.44 (d) 55.435

Soln. For 17.8, number of significant digits is 3. For 3.1143, number of significant digits is 5.

$17.8 \times 3.1143 = 55.4354$

One can only claim 3 accurate significant digits.

Therefore, $17.8 \times 3.1143 = 55.4$

Correct option is (b)

9. An experiment measures quantities *a, b, c* and *x* is calculated from $x = ab/c^3$. If the maximum percentage of error in *a, b* and *c* are 1%, 1% and 2% respectively, the maximum percentage of error in *x* will be

(JNU Biotech, 2007)

(a) 8% (b) 4% (c) –4% (d) None of these

Soln. Given : $x = \frac{ab}{c^3}$

Taking log on both sides,

$\log x = \log a + \log b + -3 \log c$

Differentiating both sides,

$$\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b} - \frac{3\Delta c}{c}$$

For error calculation, taking the maximum value

$$\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{3\Delta c}{c}$$

Multiplying by 100 on both sides, we get % error

Percentage error is x = 1% + 1% + (3 × (2%)) = 8%

Correct option is (a)

10. Which of the following is not a dimensionless quantity? **(JNU Biotech, 2011)**

(a) Strain (b) Solid angle (c) Dielectric constant (d) Planck's constant

Soln. Dimension of Planck's constant : $[ML^2T^{-1}]$

Strain, solid angle, and dielectric constants are all dimensionless quantities

Correct option is (d)

11. The following quantity has the dimension of action **(JNU Biotech, 2011)**

(a) Energy (b) Planck's constant (c) Angular momentum (d) Torque

Soln. The action is represented as

$$S = \int_{t_1}^{t_2} L dt$$

where L is the Lagrangian which has the dimensions of energy. Therefore, dimensions of action is $[ML^2T^{-2}]\ [T] = ML^2T^{-1}$, which is also the dimensions of Planck's constant.

Correct option is (b)