4

Application of Derivatives

Increasing and Decreasing Function:

A continuous function in an interval I is

(a) Strictly increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall x_1, x_2 \in I$

(b) Increasing if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)\ \forall\ x_1, x_2 \in I$

(c) Strictly decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall x_1, x_2 \in I$

(d) decreasing if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)\ \ \forall x_1, x_2 \in I$

In other way, A continuous function on interval I and differentiable on I is

(a) strictly increasing if $f'(x) > 0\ \ \forall x \in I$ (b) increasing if $f'(x) \ge 0\ \ \forall x \in I$

(c) strictly decreasing if $f'(x) < 0\ \ \forall x \in I$ (d) decreasing if $f'(x) \le 0\ \ \forall x \in I$

Ex. The function $f(x) = \sin(x) + \cos(x)$, where $0 \le x \le 2\pi$, is increasing in the interval

(a) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$ (b) $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(c) $\left(0, \frac{5\pi}{4}\right)$ (d) $\left(\frac{\pi}{4}, 2\pi\right)$

Soln. $f(x) = \sin(x) + \cos(x)$

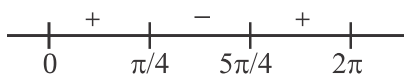
$f'(x) = \cos(x) - \sin(x)$

$f'(x) = 0 \Rightarrow \cos(x) = \sin(x)$

$\Rightarrow \tan(x) = 1$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

Sign scheme for $f'(x)$,



$f'(x) > 0\ \forall\ x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

option (a) is correct

Ex. The range of values of *x* in which $y = 2x^3 - 9x^2 + 12x + 4$ is strictly decreasing is

(a) $(-\infty, 1]$ (b) $[-1, 2]$ (c) $[1, 2]$ (d) $[-1, -2]$

Soln. $y = 2x^3 - 9x^2 + 12x + 4$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

$$= 6\left(x^2 - 3x + 2\right) = 6\left(x-2\right)\left(x-1\right)$$

$$\frac{dy}{dx} = 0 \Rightarrow x = 1, 2$$

Sign scheme for $f'(x)$

$$f'(x) < 0 \quad \forall x \in (1, 2)$$

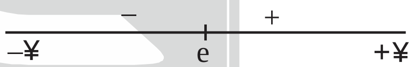
Correct option is (c)

Ex. The function $f(x) = \dfrac{x}{\ln(x)}$ is increasing in the interval

(a) (0, 1) (b) (0, e) (c) (e, ∞) (d) (0, ∞)

Soln. $f(x) = \dfrac{x}{\ln(x)}$

Now, $f'(x) = \dfrac{\ln(x)\cdot 1 - x\cdot\frac{1}{x}}{\left(\ln(x)\right)^2} = \dfrac{\ln(x) - 1}{\left(\ln(x)\right)^2}$



$$f'(x) = 0 \Rightarrow \ln(x) - 1 = 0 \Rightarrow \ln(x) = 1$$

$$\Rightarrow x = e$$

$$f'(x) > 0 \quad \forall x \in (e, \infty)$$

Option (c) is correct

Ex. The function $x + \sin(x)$ is best described as

(a) non- decreasing (b) non-increasing (c) decreasing (d) increasing

Soln. $f(x) = x + \sin(x)$

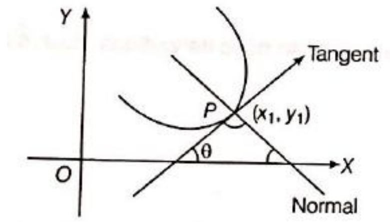
$$f'(x) = 1 + \cos(x) \geq 0 \quad \forall x \in \mathbb{R}$$

f is increasing or constant function ⇒ *f* is non- decreasing function

option (a) is correct

Slope of Tangent and Normal,

3. Let $y = f(x)$ be a continous curve and let $P(x_1, y_1)$ be point on it.



then, $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$ is the slope of tangent to the curve $y = f(x)$ at a point $P(x_1, y_1)$.

$\Rightarrow \left(\frac{dy}{dx}\right)_P = \tan\theta$ = slope of tangent at P

Where, θ is the angle which the tangent at $P(x_1, y_1)$ forms with the postive direction of X- axis as shown in the figure.

Remark:

(i) **Horizontal tangent:** if tangent is parallel to x- axis, then

$\theta = 0^\circ \Rightarrow \tan\theta = 0$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$

(ii) **Vertical tangent:** if tangent is perpendicular to x-axis or parallel to y-axis, then

$\theta = 90° \Rightarrow \tan\theta = \infty$ *or* $\cot\theta = 0$

$\therefore \left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0$

Slope of Normal:

We know that the normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passes through P.

$\therefore$ Slope of the normal at

$$\text{P} = \frac{-1}{\text{Slope of the tangent at P}}$$

$\Rightarrow$ slope of normal at $\text{P}(x_1, y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$

or slope of normal at $\text{P}(x_1, y_1) = -\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

Remark :

(i) Horizontal normal: if normal is parallel to x-axis, then

$-\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0$ or $-\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0$

(ii) Vertical normal: If normal is perpendicular to x-axis or parallel to y-axis then $-\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$

Ex. Find the slopes of the tangent and normal to the curve $x^3+3xy+y^3=2$ at $(1,1)$.

Soln. Given equation of curve is $x^3+3xy+y^3=2$

Differentiating it w.r.t. x, we get

$$3x^2+3x\frac{dy}{dx}+3y+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(3x^2+3x)}{(3y+3y^2)} \Rightarrow \frac{dy}{dx}=-\frac{(x^2+y)}{(x+y^2)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)}=-\left(\frac{2}{2}\right)=-1$$

$\therefore$ Slope of tangent at $(1,1)=\left(\frac{dy}{dx}\right)_{(1,1)}=-1$

and slope of normal at $(1,1)=-\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}}=\frac{-1}{-1}=1$

Ex. Find the point on the curve $y=x^3-3x$ at which tangent is parllel to x- axis.

Soln. Let the point at which tangent is parallel to x- axis be $P(x_1, y_1)$,

Then, it must lie on curve,

Therefore, we have $y_1=x_1^3-3x_1$

Differentiating $y=x^3-3x$ w.r.t. x, we get

$$\frac{dy}{dx}=3x^2-3 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=3x_1^2-3$$

Since, the tangent is parallel to x-axis

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=0 \Rightarrow 3x_1^2-3=0$$

$\Rightarrow x_1=\pm 1$

From Eqs. (i) and (ii), we get

When $x_1=1$, then $y_1=1-3=-2$

When $x_1=-1$, then $y_1=-1+3=2$

$\therefore$ Points at which tangent is parallel to x- axis. are (1,–2) and (–1,2)

Ex. Find the point on the curve $y=x^3-2x^2-x$ at which the tangent line is paralle to the line $y=3x-2$

Soln. Let $P(x_1, y_1)$ be the required point.

Then we have $y_1=x_1^3-2x_1^2-x_1$ ..(i)

Differentiating the caurve $y=x^3-2x^2-x$ w.r.t x, we get

$$\frac{dy}{dx}=3x^2-4x-1 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=3x_1^2-4x_1-1$$

Since, tangent at (x_1, y_1) is parallel ton the line $y=3x-2$

$\therefore$ slope of the tangent at $P(x_1, y_1)$ = Slope of the line

$y = 3x - 2 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3$

$\Rightarrow 3x_1^2 - 4x_1 - 1 = 3 \Rightarrow 3x_1^2 - 4x_1 - 4 = 0$

$\Rightarrow (x_1 - 2)(3x_1 + 2) = 0 \Rightarrow x_1 = 2, -2/3$ ...(ii)

From Eqs. (i) and (ii), we get

When $x_1 = 2,$ then

$y_1 = 8 - 8 - 2 \Rightarrow y_1 = -2$

When $x_1 = -2/3$, then $y_1 = x_1^3 - 2x^2 - x_1$

$\Rightarrow y_1 = \frac{-8}{27} - \frac{8}{9} + \frac{2}{3} \Rightarrow y_1 = \frac{-14}{27}$

Thus, the point at which tangent is parallel to $y = 3x - 2$ are $(2, -2)$ and $\left(-\frac{2}{3}, \frac{-14}{27}\right)$

Maxima and Minima

Local maximum/Local Minimum

A point c in the interior of the domain of *f* is called

(a) Local maxima, if there exists an $h > 0$ such that $f(c) > f(x)\ \forall\ x \in (c - h, c + h)$.

The value *f*(c) is called the local maximum value of *f*.

(b) local minima, if there exists $h > 0$ such that

$f(c) < f(x)\ \forall\ x \in (c - h, c + h)$

The value *f*(c) is called the local minimum value of *f*.

Absolute Maximum/Absolute Minimum

A function *f* is defined by $[a, b]$ is said to be,

(a) absolute maximum at $x = c \in [a, b]$ if $f(x) \le f(c)\ \forall x \in [a, b]$

(b) absolute minimum at $x = d \in [a, b]$ if $f(x) \ge f(d)\ \forall x \in [a, b]$

Stationary points and Stationary values of a function

Definition: If $f'(c) = 0$, then $x = c$ is called a stationary point of *f* and *f*(*c*) is called stationay value of *f*.

Critical points and ciritical values of a function

Definition: A point $x = c$ such that either $f'(c)$ does not exist or $f'(c) = 0$ is called a critical point of *f* and $f(c)$ is called a critical value of *f*.

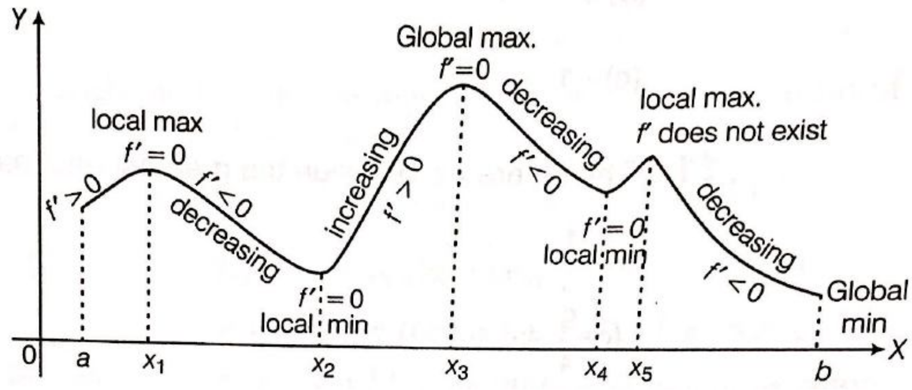
Second derivative Test : If f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. $f(c)$ is called local maximum value.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. *f*(c) is called local minimum value.

(iii) The test faits if $f'(c)=0$ and $f''(c)=0$. we back to first derivative test.

Base on the above discussion we can summarize things in a single graph as given below:



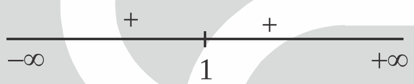
Ex. For the function $y = x^3 - 3x^2 + 3x + 1,$ the point $x = 1$ is

(a) minima (b) maxima (c) an inflection point (d) None of these

Soln. $y = x^3 - 3x^2 + 3x + 1$

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3\left(x^2 + 2x + 1\right) = 3\left(x - 1\right)^2$$

$$\frac{d^2y}{dx^2} = 6x - 6$$



At $x = 1$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$

Sign scheme of $f'(x)$

$f'(x)$ does not change its sign at $x = 1$

$x = 1$ is an inflection point

Option (c) is correct

Ex. At what points does the graph of $y = 2x^3 - 6x^2$ equally reach a maximum and a minimum

(a) (0,0) (maximum) and (2,8) (minimum) (b) (0,0) (maximum) and (2,–8) (minimum)

(c) (0,0) (maximum) and (3,–8) (minimum) (d) (2,–8) (maximum) and (0,0) (minimum)

Soln. $y = 2x^3 - 6x^2$

$$\frac{dy}{dx} = 6x^2 - 12x = 6x\left(x - 2\right)$$

$$\frac{d^2y}{dx^2} = 12x - 12 = 12\left(x - 1\right)$$

for critical points $\frac{dy}{dx} = 0 \Rightarrow x = 0, 2$

At $x = 0 \Rightarrow \frac{d^2y}{dx^2} < 0 \Rightarrow$ local maxima

and At $x = 2, \frac{d^2y}{dx^2} > 0 \Rightarrow$ local minima

$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$ (maximum)

$x = 2 \Rightarrow y = -8 \Rightarrow (2, -8)$ (minimum)

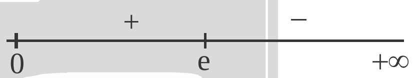
option (b) is correct

Ex. The function $f(x) = \frac{\ln(x)}{x}$, has a maximum at

(a) $x = 1$ (b) $x = \ln(2)$ (c) $x = e$ (d) $x = 1/e$

Soln. $f(x) = \frac{\ln(x)}{x}, x > 0$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$$



$f'(x) = 0 \Rightarrow 1 - \ln(x) = 0$

$\Rightarrow \ln(x) = 1$

$\Rightarrow x = e$

Sign scheme for $f'(x)$

f' is changing sign from + ve to –ve at $x = e$

$\therefore f$ has local maxima at $x = e$

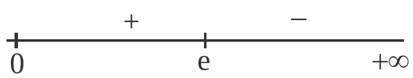
Option (c) is correct

Ex. The maximum value of the function $f(x) = \frac{\ln(x)}{x}, x > 0$ is equal to

(a) 0 (b) e (c) $\frac{1}{e}$ (d) $\frac{2}{e}$

Soln. $f(x) = \frac{\ln(x)}{x}$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$$



$f'(x) = \Rightarrow 1 - \ln(x) = 0$

$\Rightarrow x = e$

Sign sechme for $f'(x)$

$f'(x)$ is changing sign from +ve to –ve at $x = e$

$\Rightarrow x = e$ is point of maxima

So, maximum value of f at $x = e$ is,

$$f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$$

Option (c) is correct

Ex. If f is continuous on $[a, b]$ and $\frac{df}{dx} = 0$ for every $x \in (a,b)$, then f is

(a) strictly increasing function on $[a, b]$
(b) strictly decreasing function on $[a,b]$
(c) constant function on $[a,b]$
(d) None of these

Soln. $\frac{df}{dx} = 0 \ \forall x \in (a,b) \Rightarrow \ f = k \ \forall x \in (a,b)$

where k is constant

$\Rightarrow f$ is a constant function on $[a, b]$

Correct option is (c)

Ex. Consider the function $f(x) = Ax^4 - Bx^2$ such that $A > 0$ and $B > 0$. Which of the following statements are true ?

P : The function has a maxima at $x = 0$

Q : The function has a minima at $x = 0$

R : The value of the function is zero at $x = 0$

S : The value of the function is non-zero at $x = 0$

(a) Only Q and S (b) Only P and R (c) Only P and S (d) Only Q and R

Soln. $f(x) = Ax^4 - Bx^2,\ A > 0,\ B > 0$

$$f'(x) = 4Ax^3 - 2Bx$$

$$f''(x) = 12Ax^2 - 2B$$

for critical points, $f'(x) = 0$

$$\Rightarrow 4Ax^3 - 2Bx = 0$$

$$\Rightarrow x(4Ax^2 - 2B) = 0$$

$$\Rightarrow x = 0 \ \ or \ \ x^2 = \frac{2B}{4A} = \frac{B}{2A}$$

$$\Rightarrow x = 0 \ \ or \ \ x = \pm\sqrt{\frac{B}{2A}}$$

$$f''(0) = -2B < 0$$

f has local maxima at $x = 0$