$$\Rightarrow \int_{\ell/4}^{3\ell/4}\sqrt{\frac{2}{\ell}}\sin\frac{n\pi x}{\ell}\sqrt{\frac{2}{\ell}}\sin\frac{n\pi x}{\ell}$$

$$\Rightarrow \frac{2}{\ell}\int_{\ell/4}^{3\ell/4}\sin^2\frac{n\pi x}{\ell} \quad \Rightarrow \quad \frac{n\pi}{\ell}=a$$

$$\Rightarrow \frac{2}{\ell}\int_{\ell/4}^{3\ell/4}\sin 2ax \quad \Rightarrow \quad \frac{2}{\ell}\int_{\ell/4}^{3\ell/4}\left[\frac{1-\cos 2ax}{2}\right]dx$$

$$\Rightarrow \frac{2}{2\ell}\left[\int_{\ell/4}^{3\ell/4}1dx-\int_{\ell/4}^{3\ell/4}\cos 2axdx\right] \quad \Rightarrow \quad \frac{1}{\ell}\left[[x]_{\ell/4}^{3\ell/4}-\left[\frac{\sin 2ax}{2a}\right]_{\ell/4}^{3\ell/4}\right]$$

$$\Rightarrow \frac{1}{\ell}\left[\left(\frac{3\ell}{4}-\frac{\ell}{4}\right)-\frac{1}{2a}\left(\sin 2a\frac{3\ell}{4}-\sin\frac{2a.\ell}{2}\right)\right]$$

$$\Rightarrow \frac{1}{\ell}\left[\frac{\ell}{4}-\frac{1}{2a}\left(\sin\frac{3a\ell}{2}-\sin\frac{a\ell}{2}\right)\right] \quad \Rightarrow \quad \frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{2a}\left(\sin\frac{n\pi}{4}\times\ell-\sin\frac{n\pi}{\ell}\frac{\ell}{2}\right)\right]$$

$$\Rightarrow \frac{1}{\ell}\left[\frac{1}{\ell}-\frac{1}{2a}\left(\sin\frac{3n\pi}{2}-\sin\frac{n\pi}{2}\right)\right]$$ [For ground state n = 1]

$$\Rightarrow \frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{2a}\left(\sin\frac{3\pi}{2}-\sin\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{2a}\left(\sin 270^\circ-\sin 90^\circ\right)\right] \quad \left[\sin 270^\circ=\sin\left(180^\circ+90^\circ\right)=-\sin 90^\circ=-1\right]$$

$$\Rightarrow \frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{2a}\left(-\sin 90^\circ-\sin 90^\circ\right)\right]$$

$$\Rightarrow \frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{2a}\left(-(+1)-1\right)\right]\Rightarrow\frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{2a}(-1-1)\right]\Rightarrow\frac{1}{\ell}\left[\frac{\ell}{2}-\frac{1}{\frac{n\pi}{\ell}}(-2)\right]\Rightarrow\frac{1}{\ell}\times\frac{\ell}{2}+\frac{1}{\ell}.\frac{\ell}{n\pi}=\frac{1}{2}+\frac{1}{\pi}$$

Correct answer is (b)

5. A wave funtion given by $\psi=\sin\frac{n\pi x}{\ell}\left(0\le x\le\ell\right)$ is it normalised ? If not, normalise it.

Soln. $\int_0^\ell\psi^*\psi\,dx=\int_0^\ell\sin^2\frac{n\pi x}{\ell}\,dx=\frac{\ell}{2} \quad \because \quad \int_0^\ell\psi^*\psi\,dx\ne 1$

So, the function is not normalised.

To normalise this wave function, a constant A is taken with the function ψ and now function becomes

$(A\psi)=\psi'$

So, according to the condition of normalisation,

$$\int_0^\ell\psi'\psi'd\tau=1 \Rightarrow \int_0^\ell\psi'^2\,dx=1 \Rightarrow \int_0^\ell A^2\psi^2dx=1$$

$$\Rightarrow A = \sqrt{\frac{1}{\int_0^{\ell} \psi^2 dx}} = \sqrt{\frac{1}{\ell/2}} = \sqrt{\frac{2}{\ell}}$$

Therefore, normalised function = $= \sqrt{\frac{2}{\ell}} \sin \frac{n\pi x}{\ell}$

6. A particle is confined -*a* to +*a* and gives a wavefunction $\cos \frac{\pi x}{2a}$. Is it normalised? If not, normalise it.

$$\int_{-a}^{+a} \psi^* \psi \, d\tau = \int_{-a}^{+a} \cos\frac{\pi x}{2a} \cos\frac{\pi x}{2a} dx$$

$$= \int_{-a}^{+a} \cos^2 \frac{\pi x}{2a} dx \qquad \left[\int \cos^2 \theta = \frac{1+\cos 2\theta}{2} \right]$$

$$= \int_{-a}^{+a} \frac{1+\cos\frac{\pi x}{a}}{2} dx = \frac{1}{2}\int_{-a}^{+a} dx + \frac{1}{2}\int_{-a}^{+a} \cos\frac{\pi x}{a} dx = \frac{1}{2}[x]_{-a}^{+a} + \frac{1}{2}\frac{a}{\pi}\left[\sin\frac{\pi x}{a}\right]_{-a}^{+a}$$

$$= \frac{1}{2}\left[a-(-a)\right] + \frac{1}{2}\frac{a}{\pi}\left[\sin\frac{\pi a}{a} - \sin\frac{\pi(-a)}{a}\right] = \frac{1}{2}2a + \frac{a}{2\pi}\left[\sin\pi - \sin(-\pi)\right]$$

$$= a + \frac{a}{2\pi}\left[\sin\pi + \sin\pi\right] \qquad \left[\because \ \sin(-x) = -\sin x\right]$$

$$= a + \frac{2a\sin\pi}{2\pi} = a + 0 = a$$

Therefore, function is not normalised.

To be normalised, $\psi' = A\,\psi$

$$\therefore \int_{-a}^{+a} A\psi^* \, A\psi \, dx = 1 \Rightarrow A^2 \int_{-a}^{+a} \psi^*\psi \, dx = 1 \Rightarrow A = \sqrt{\frac{1}{\int_{-a}^{+a} \psi^* \psi \, dx}} = \sqrt{\frac{1}{a}}$$

Normalised wave function $= \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a}$.

7. Which of the following functions are 'well behaved' quantum mechanically?

(a) $\exp(-ax^2)$ (b) $\exp(-ax)$ (c) x^2 (d) x

Soln. For well behaved function, function must vanish at $\pm\infty$.

$\exp(-ax^2)$ at $x = \infty$ $\exp(-a\infty^2) = 0$

at $x = -\infty$ $\exp(-a\infty^2) = 0$

Correct option is (a)