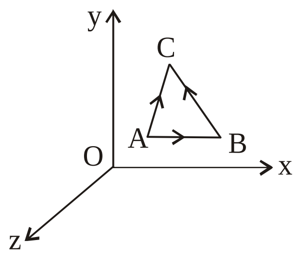
KINEMATICS (MOTION IN 1-D AND 2-D)

DISTANCE AND DISPLACEMENT:

Consider an example where a particle moves from A to B and then B to C and stops, as shown in figure.



The total length of path traversed AB + BC is called the distance. The length between A and C, i.e., AC is the length between the initial and final position. This length is called displacement.

- Distance only has magnitude and is always a positive quantity.
  It is a scalar quantity.
- Displacement has both magnitude and direction.
  Hence, it is a vector quantity. Displacement add according to the triangle rule of vector addition.
- Dimension of distance and displacement is length [L]. SI unit is meter (m).

AVERAGE SPEED AND INSTANTANEOUS SPEED:

Average Speed: It is defined as the distance travelled in unit time interval.

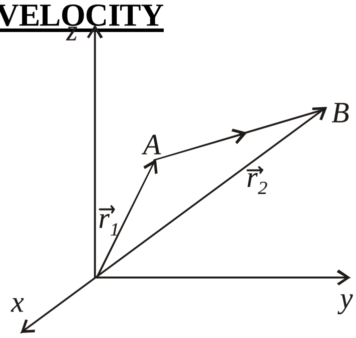
$$v_{av} = \frac{s}{t_f - t_i}$$ where s is the distance travelled in time t_i (initial time) to t_f (final time).

Instantaneous Speed: It is defined at a time as $v = \lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$ where s is the distance travelled in time t.

- Average speed is defined for a time interval
- Instantaneous speed is defined at a particular instant. It is also called speed.

AVERAGE VELOCITY AND INSTANTANEOUS VELOCITY

Average Velocity: It is defined as the displacement divided by the time interval $\left(t_i \text{ to } t_f\right)$



$$\vec{v}_{av} = \frac{\overrightarrow{AB}}{t_f - t_i} = \frac{\vec{r}_2 - \vec{r}_1}{t_f - t_i}$$

- Average velocity, like displacement, is a vector.

Instantaneous Velocity: It is defined as $\vec{v} = \lim\limits_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$

Magnitude of velocity

$$v = \left|\frac{d\vec{r}}{dt}\right| = \frac{|d\vec{r}|}{dt} = \frac{ds}{dt} \equiv \text{ instantaneous speed.}$$

- Instantaneous velocity is called velocity.

AVERAGE ACCELERATION AND INSTANTANEOUS ACCELERATION

Average Acceleration: It is defined as the change in velocity divided by the time interval.

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Where, $\vec{v}_2$ and $\vec{v}_1$ are velocities at t_2 and t_1, respectively.

Instantaneous Acceleration: It is defined as $\vec{a} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{v}}{dt} = \dfrac{d\vec{v}}{dt}$

- Instantaneous acceleration is also called acceleration.
- Dimension of acceleration is LT^{-2} and its S.I. unit is metre/second^{-2} (m/s^2).

MOTION IN 1-D (MOTION IN A STRAIGHT LINE)

Let us assume a particle moves along a straight line (say, x-axis).
Also, let the particle be at origin at time $t = 0$. Then the position of the particle at time t is given by its coordinate x at that time. The velocity of the particle is given by $v = \dfrac{dx}{dt}$

and its acceleration is $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

- If $\dfrac{dx}{dt} > 0$, the direction of v is along positive x-axis, and if $\dfrac{dx}{dt} < 0$, the direction, is along negative x-axis.
- If $\dfrac{dv}{dt} > 0$, the direction of a is along positive x-axis and if $\dfrac{dv}{dt} < 0$, the direction, is along negative x-axis.
- If velocity and acceleration are both positive and are both negative, the speed increases.
- If velocity and acceleration have opposite signs, the speed decreases. The particle is said to be decelerating.

Motion with Constant Acceleration

Let us suppose a particle moves with constant acceleration a and has initial velocity u at $t = 0$. Its velocity at time t be v.

$\dfrac{dv}{dt} = a$ or, $dv = a\,dt$

or, $\displaystyle\int_u^v dv = \int_0^t a\,dt \Rightarrow v - u = at \Rightarrow v = u + at$ ...(i)

Using (i), we get,

$$v = \frac{dx}{dt} = u + at \text{ or, } \int_0^x dx = \int_0^t (u + at)\,dt \quad [\text{where } x \text{ is position at } t]$$

$$\text{or, } x = ut + \frac{1}{2}at^2 \qquad \text{...(ii)}$$

Using, $\frac{dv}{dt} = a$

or, $\frac{dv}{dx}\frac{dx}{dt} = a$

or, $v\frac{dv}{dx} = a \qquad \left(\because \frac{dx}{at} = v\right)$

or, $\int_u^v v\,dv = \int_o^x a\,dx$

or, $\left.\frac{v^2}{2}\right|_0^v = a\,x|_0^x \Rightarrow v^2 - u^2 = 2ax$

Writing all the three equations together,

$$v = u + at \;;\; x = ut + \frac{1}{2}at^2 \;;\; v^2 = u^2 + 2ax$$

- Here v, u, a and x may be negative or positive depending on their direction.
- If initial position is x_1 at $t = 0$ and x_2 at any time t, then the equations will be,

$v = u + at;\; x_2 - x_1 = ut + \frac{1}{2}at^2$ & $v^2 = u^2 + 2a(x_2 - x_1)$, where $x_2 - x_1$ is displacement.

MOTION IN 2-D DIMENSION (MOTION IN A PLANE)

The three equation derived for 1-D motion can be used here for the components of the vectors - displacement, velocity and acceleration.

The idea is that a 2-D motion can be treated as two independent 1-D motion, one along x-axis and the other along y-axis. For x-component we can write,

$$v_x = u_x + a_x t \;;\; x = u_x t + \frac{1}{2}a_x t^2 \;;\; v_x^2 = u_x^2 + 2a_x x$$

Similarly for y-component,

$$v_y = u_y + a_y t \;;\; y = u_y + \frac{1}{2}a_y t^2 \;;\; v_y^2 = u_y^2 + 2a_y y$$

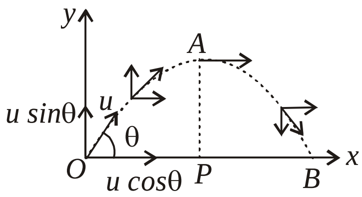
PROJECTILE MOTION:

It is an example of 2-D motion (motion in a plane) with constant acceleration. If a particle is thrown at an angle with horizontal near the earth's surface, it moves in curved path.

The particle is called a projectile and its motion is called projectile motion.

Assumptions in Projectile Motion:

- The projectile remains close to the surface of the earth such that the value of acceleration due to gravity ($g \approx 9.8\ m/s^2$) remains constant.
- The air resistance on the projectile during the motion is assumed to be negligible.

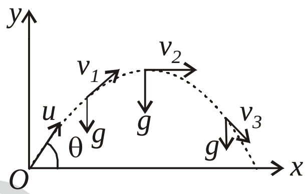


Various Terms Associated With Projectile Motion are as Follows:

- Point O is called point of projection.
- θ is called the angle of projection.
- OB is horizontal range or range.
- AP is maximum vertical range attained also known as maximum height.
- The total time taken by the particle to transverse the projectile trajectory OAB is called time of flight.

  The acceleration due to gravity always acts downwards along negative *y*-axis. One can split the velocity and acceleration into components along *x* and *y* axes:

  $u_x = u\cos\theta$, $u_y = u\sin\theta$, $a_x = 0$, $a_y = -g$



Sign Convention Used in Projectile Motion:

- If a component is along positive *y*-direction or positive *x*-direction then it is taken to be positive.
- If a component is along negative *y*-direction or negative *x*-direction, then it is taken to be negative.

  Now, one can solve two independent 1-D motion separately.

Motion Along *x*-axis (Horizontal Motion)

Acceleration: $a_x = 0$

Velocity at any time *t*: $v_x = u_x + at = u_x \Rightarrow v_x = u\cos\theta$

Displacement: $x = u_x t + \frac{1}{2}a_x t^2 = u_x t \Rightarrow x = ut\cos\theta$

Velocity along x-axis remains constant throughout the motion.

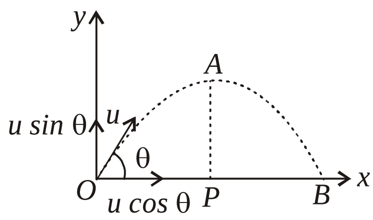
Motion Along y-axis (Vertical Motion)

Acceleration: $a_y = -g$

Velocity: $v_y = u_y + a_y t = u_y - gt$

Displacement: $y = u_y t - \frac{1}{2}gt^2$

Calculation of Time of Flight



The particle is at B, say, at time t shown in figure.

$OB = x = ut\cos\theta$

The displacement of projectile along y is zero at B.

Using the vertical motion equation,

$$y = u_y t - \frac{1}{2}gt^2 \text{ or, } 0 = ut\sin\theta - \frac{1}{2}gt^2 \Rightarrow t\left(u\sin\theta - \frac{1}{2}gt\right) = 0$$

Thus, $y = 0$ at $t = 0$ and $t = \dfrac{2u\sin\theta}{g}$

Now, at $t = 0$, $y = 0$, corresponds to point O. Therefore, $t = \dfrac{2u\sin\theta}{g}$ is at B.

$$T = \frac{2u\sin\theta}{g} \text{ (Time of flight)}$$

Horizontal Range

Displacement along x-direction is equal to the distance travelled by particle in time of flight, $T = \dfrac{2u\sin\theta}{g}$ or,

$$x = ut\cos\theta = u\cos\theta\frac{2u\sin\theta}{g} = \frac{u^2\sin 2\theta}{g} \text{ or } R = \frac{u^2\sin 2\theta}{g}$$

Maximum Height

The vertical height AP is the maximum height. At this height vertical component of velocity becomes zero.

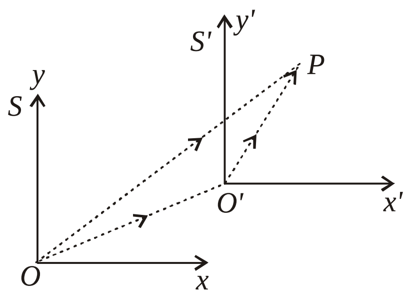
Using equation for vertical motion,

$$v_y^2 - u_y^2 = -2gy$$

or, $u_y^2 = 2gH$ (At $y = h, v_y = 0$) or, $H = \dfrac{u^2\sin^2\theta}{2g}$

RELATIVE VELOCITY AND CHANGE OF FRAME

Consider two frames of reference S and S' and a particle, P is observed from both the frames. The two frames may be moving with respect to each other.



$\vec{r}_{PS} = \overrightarrow{OP}$ - position vector of P w.r.t. S frame

$\vec{r}_{PS'} = \overrightarrow{O'P}$ - position vector of P w.r.t. S' frame

From the figure, $\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$

or, $\vec{r}_{PS} = \vec{r}_{S'S} + \vec{r}_{PS'}$ ....(i)

Here $\vec{r}_{S'S}$ is the position vector of S' frame w.r.t. S frame

Differentiating (i) w.r.t. time,

$$\frac{d}{dt}(\vec{r}_{PS}) = \frac{d}{dt}(\vec{r}_{S'S}) + \frac{d}{dt}(\vec{r}_{PS'}) \text{ or } \vec{v}_{PS} = \vec{v}_{S'S} + \vec{v}_{PS'}$$

Where $\vec{v}_{PS}$ is velocity of particle w.r.t. S, $\vec{v}_{S'S}$ is velocity of S' frame w.r.t S, and $\vec{v}_{PS}$, is velocity of particle w.r.t. S'.

- It is assumed that time has same meaning in both frames, however, it is not true if the velocity of one of the frames is comparable to speed of light.
- Equation may be written as, $\vec{v}_{PS'} = \vec{v}_{PS} - \vec{v}_{S'S}$. This implies that if the velocity of two bodies are known with respect to a common frame, one can find the velocity of one body w.r.t. O the other body.
- Relative velocity of body 1 w.r.t. body 2 is $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$

SOLVED EXAMPLES

1. A river of width 0.2 km flows with uniform speed of 1 km/hr from west to east. A boat sets off from a point S on the south bank and wishes to land at the exact opposite point N on the north bank. It can travel at a speed of 2 km/hr relative to the water. In what direction should it point in order to arrive at point N by a straight line route and how long will it take ? **[JNU Life Sc. 2004]**

(a) At an angle of 60º towards the west and it will take approximately 7 minutes
(b) At an angle of 60º towards the east and it will take approximately 4 minutes
(c) At an angle of 30º towards the west and it will take approximately 7 minutes
(d) At an angle of 30º towards the east and it will take approximately 4 minutes

Soln. Let $\vec{v}_{br}$ be the velocity of boat with respect to river, $\vec{v}_{rg}$ be the velocity of river with respect to ground and $\vec{v}_{bg}$ is the required velocity of boat with respect to the ground. Let the boat be directed at an angle θ as shown in figure. From figure, we have $\vec{v}_{bg} = \vec{v}_{br} + \vec{v}_{rg}$. Equating the components along x-axis,

$$0 = -v_{br} \sin\theta + v_{rg}$$

or, $0 = -2\sin\theta + 1 \Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = 30°$

Equating component along y-axis

$$v_{bg} = v_{br}\cos\theta = 2\cos 30° = \sqrt{3}\ \text{km/hr}$$

$$\text{Time taken to cross the river} = \frac{\text{width}}{\text{velocity}} = \frac{0.2\ \text{km}}{\sqrt{3}\ \text{km/hr}} = \frac{0.2\times 60}{\sqrt{3}}\ \text{min}$$

$$\simeq 6.928 \text{ minutes} \ \simeq 7 \text{ minutes}$$

Therefore, the boat should be directed at an angle of 30° towards west and it will take approximately 7 minutes.

Correct option is (c).

2. A helicopter on a flood relief mission flying horizontally to the ground with a speed of u at an altitude H has to drop a food packet for a victim standing on the ground. Assuming that the victim stands in the vertical plane of motion of the helicopter, at which distance from the victim should the food packet be dropped ? **[JNU Life Sc. 2006]**

(a) $u\sqrt{\left(\dfrac{2H}{g}\right)}$ (b) $\dfrac{2Hu^2}{g}$ (c) $\sqrt{\left\{\left(\dfrac{2u^2H}{g}\right) + H^2\right\}}$ (d) $\sqrt{(2H)}$