Singular Matrix: A square matrix is a singular matrix if its determinant is zero.

i.e. $|A| = 0$ then A is singular matrix but if $|A| \neq 0$ then it is called non-singular matrix.

Inverse of Matrix: The inverse of a matrix A exists iff A is non-singular (i.e. $|A| \neq 0$) then

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Properties of Inverse:

(1) $AA^{-1} = A^{-1}A = I$

(2) If A and B are inverse of each other i.e. $AB = BA = I$ (If inverse of A exist then A is called invertible)

(3) $(AB)^{-1} = B^{-1}A^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(5) If $|A| \neq 0$ then $(A^T)^{-1} = (A^{-1})^T$, $(A^{-1})^\theta = (A^\theta)^{-1}$

(6) If $|A| \neq 0$, $|A^{-1}| = |A|^{-1}$ i.e. $AA^{-1} = I \Rightarrow |AA^{-1}| = I \Rightarrow |A|\ |A^{-1}| = I \Rightarrow |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

Properties of Adjoint:

(1) $(\text{adj } A)\, A = |A|\, I_n = A(\text{adj } A)$

(2) $$A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & 0 \\ 0 & |A| & 0 & 0 \\ 0 & 0 & |A| & 0 \\ 0 & 0 & 0 & |A| \end{bmatrix} = |A|\ I_n$$

(3) $|A \text{ adj } (A)| = |A|^n$

(4) $|\text{adj } (A)| = |A|^{n-1}$, $A(\text{adj } A) = |A|\, I_n \Rightarrow |A|\ |\text{adj } A| = |A|^n \Rightarrow |\text{adj } A| = |A|^{n-1}$, provided $|A| \neq 0$

(5) $\text{adj}(AB) = (\text{adj } B)\,(\text{adj } (A))$

(6) $\text{adj } A^T = (\text{adj } A)^T$

(7) $\text{adj } (\text{adj } A) = |A|^{n-2} A$

Let $\text{adj } A = B$ ($\because B(\text{adj } B) = |B|\, I_n$) $\Rightarrow (\text{adj } A)(\text{adj } (\text{adj } A)) = |\text{adj } A|\, I_n$

$\Rightarrow (\text{adj } A)(\text{adj } (\text{adj } A)) = |A|^{n-1} \Rightarrow A(\text{adj } A)\, \text{adj } (\text{adj } A) = |A|^{n-1} A$

$\Rightarrow |A|\, \text{adj } (\text{adj } A) = |A|^{n-1} A \Rightarrow \text{adj } (\text{adj } A) = |A|^{n-2} A$

Properties of Determinant:

(1) $|A| = |A^T|$

(2) If any row (or column) of a matrix is zero then $|A| = 0$

i.e. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $|A| = 0$

(3) If $A = [a_{ij}]$ is diagonal matrix of order n (> 2) then

$|A| = a_{11}\, a_{22}\, a_{33}\, ...\, a_{nn}$

i.e. $A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \Rightarrow |A| = 1.2.3 = 6$

(4) If A and B are square matrix of same order then

$|AB| = |A||B|$

(5) The sum of product of element of any row (or columns) with their cofactors is always equal to $|A|$

i.e. $\sum_{j=1}^{n} a_{ij}\, c_{ij} = |A|$ $\quad\sum_{i=1}^{n} a_{ij}\, c_{ij} = |A|$

(6) The sum of product of element of any row (or columns) of matrix A with the corresponding elements of some other row (column) of cofactor matrix A is zero.

$\sum_{j=1}^{n} a_{ij}\, c_{kj} = 0, i \neq k$ $\quad\sum_{i=0}^{n} a_{ij}\, c_{ik} = 0, j \neq k$

i.e. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, cofactor matrix = $\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

then $|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$

$|A| = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$

$0 = a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23}$ etc.

(7) Let $A = [a_{ij}]$ be square matrix of $n(>, 2)$ and B be matrix obtained by interchanging any two rows or columns of A then $|B| = -|A|$

i.e. $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, $|B| = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \Rightarrow |B| = -|A|$

(8) $|KA| = K^n\,|A|$

i.e. $\begin{vmatrix} Ka_1 & Ka_2 & Ka_3 \\ Kb_1 & Kb_2 & Kb_3 \\ Kc_1 & Kc_2 & Kc_3 \end{vmatrix} = K^3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\begin{vmatrix} Ka_1 & Ka_2 & Ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(9) $$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(10) If any two rows (or columns) are identical, then $|A| = 0$

i.e. $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \Rightarrow |A| = 0$

(11) Let A be square matrix and B be the matrix obtained from A by adding to row (or column) of A by scalar multiple of another row of A. Then

$|B| = |A|$

i.e. $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad |B| = \begin{vmatrix} a_1+kc_1 & a_2+kc_2 & a_3+kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + k\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |A| + k.0 = |A|$

Rank of Matrix:

Rank is defined for any matrix $(m \times n)$

A number r is said to be rank of a matrix $(m \times n)$ iff

(1) There is atleast one square sub-matrix of A of order r whose determinant is not equal to zero.

(2) If matrix A contain any square sub-matrix of order $(r + 1)$ then its determinant is zero.

Properties of Rank:

(1) $r(A^T) = r(A)$

(2) $r(AB) \le r(A)$, $r(AB) \le r(B)$ and $r(AB) \le \min\{r(A), r(B)\}$

(3) Rank of sum of two matrices cannot exceed the sum of their rank

$r(A + B) \le r(A) + r(B)$

(4) Nullity $n(A) = n - r(A)$ (Rank - Nullity theorem

Here A is square matrix, $r(A)$ is rank, n is the number of rows of matrix

(5) $r(AB) \ge r(A) + r(B) - n$, for any $a \times n$ matrix A and $n \times b$ matrix B

Eg. (1) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

$|A| = 2\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} - 1\begin{vmatrix} 0 & -2 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 2(-9 + 8) - 1(4) - 1(-6) = -2 - 4 + 6 = 0$

but $\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \neq 0$ i.e. $r(A) = 2$.

Solved Problems

1. Let $\alpha = e^{2\pi i/5}$ and the matrix

$$M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}.$$

Then, the trace of the matrix $I + M + M^2$ is

(a) –5 (b) 0 (c) 3 (d) 5

[GATE-2012]

Soln. We have $\alpha = e^{\frac{2\pi i}{5}} = (-1)^{2/5} = 1^{1/5} \Rightarrow \alpha^5 = 1$

i.e., α is fifth root of unity.

$\therefore \; 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$ ...(i)

and $1 \cdot \alpha \cdot \alpha^2 \cdot \alpha^3 \cdot \alpha^4 = 1$ ...(ii)

$$\text{Now, } M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$$

$\therefore$ trace of $(M) = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$$M^2 = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1 & \alpha + \alpha^2 & \alpha^2 + \alpha^3 + \alpha^4 & \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 & \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 \\ 0 & \alpha^2 & \alpha^3 + \alpha^4 & \alpha^5 + \alpha^5 + \alpha^6 & \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 \\ 0 & 0 & \alpha^4 & \alpha^5 + \alpha^6 & \alpha^6 + \alpha^7 + \alpha^8 \\ 0 & 0 & 0 & \alpha^6 & \alpha^7 + \alpha^8 \\ 0 & 0 & 0 & 0 & \alpha^8 \end{bmatrix}$$

$$\text{Trace}\,(M^2) = 1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 = \frac{1\left[1 - (\alpha^2)^5\right]}{1 - \alpha^2} = \frac{1 - e^{4\pi i}}{1 - e^{4\pi i/5}} \qquad \left(\because \; \alpha = e^{2\pi i/5}\right)$$

$$= \frac{1 - 1}{1 - e^{4\pi i/5}} = 0$$

Also trace $(I)_{5 \times 5} = 5$

Hence, trace $(I + M + M^2)$

= trace (I) + trace (M) + trace (M^2) = 5 + 0 + 0 = 5.

Correct option is (d)