Example: The set of concentric circles is defined by $x^2 + y^2 = c$ is a one parameter family if c takes all positive real values.

There will be n arbitary constants in n-parameter family of curves. Then, we can obtain n^{th} order differential equation whose solution is the given family.

Example 1: The differential equation of all circles passing through the origin and having their centres on the y-axis

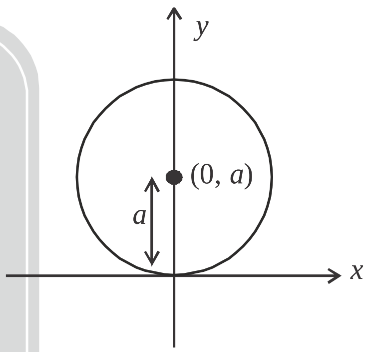
(a) $\left(x^2+y^2\right)dy/dx = 2xy$ (b) $\left(x^2-y^2\right)dy/dx = 2xy$

(c) $dy/dx = 2xy\left(x^2+y^2\right)$ (d) $dy/dx = 2xy\left(x^2-y^2\right)$

Soln: Equation of the circle passing through the origin and having their centers on the y-axis will be of the form

$$x^2+(y-a)^2 = a^2$$

$$\Rightarrow 2x+2(y-a)\frac{dy}{dx} = 0 \Rightarrow (y-a) = -\frac{x}{(dy/dx)} \Rightarrow a = y+\frac{x}{(dy/dx)}$$



Therefore, $x^2+\dfrac{x^2}{(dy/dx)^2} = \left(y+\dfrac{x}{(dy/dx)}\right)^2$

$$\Rightarrow x^2 = y^2+\frac{2xy}{(dy/dx)} \Rightarrow \left(x^2-y^2\right)\frac{dy}{dx} = 2xy$$

Correct option is (b)

Example 2: The function $y = \ln\left[sin(x+a)\right]+b$ is the solution of the following differential equation:

(a) $y'' = -\left[1+(y')^2\right]$ (b) $y'' = y^2-(y')^2$ (c) $y'' = \left[1+(y')^2\right]$ (d) $y'' = y'+y^2$

Soln: $y = \ln\left[\sin(x+a)\right]+b \Rightarrow \dfrac{dy}{dx} = \cot(x+a) \Rightarrow \dfrac{d^2y}{dx^2} = -\operatorname{cosec}^2(x+a) = -\left[1+\cot^2(x+a)\right]$

Therefore, $\dfrac{d^2y}{dx^2} = -\left[1+\left(\dfrac{dy}{dx}\right)^2\right]$

Correct option is (a)

3.2 Differential Equations of the First Order and First Degree:

CASE I: Separation of Variables

Every Differential equation can be written in the form,

$$f(y)dy = \phi(x)dx \Rightarrow \int f(y)dy = \int \phi(x)dx + C$$

Example 3: Solve $e^{dy/dx} = (x+1)$; given $y = 3\ at\ x = 0$

Soln: Taking log of both sides we get, $\frac{dy}{dx} = \ln(x+1) \Rightarrow dy = \ln(x+1)dx$

on integration, $\int dy = \int 1.\ln(x+1)dx$

$\Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{(x+1)}dx + C \quad \Rightarrow \quad y = x\ln(x+1) - \int \frac{(x+1)-1}{(x+1)}dx + C$

$\Rightarrow \quad y = x\ln(x+1) - \int \frac{(x+1)}{(x+1)}dx + \int \frac{1}{(x+1)}dx + C \quad = x\ln(x+1) - x + \ln(x+1) + C$

$\Rightarrow \quad y = (x+1)\ln(x+1) - x + C$

Given: at $x = 0 \quad y = 3 \quad \Rightarrow \quad C = 3$

Therefore, $y = (x+1)\ln(x+1) - x + 3$

Example 4: Solve $\frac{dy}{dx} = e^y\left(e^x + x^2 e^{x^3}\right)$

Soln: $\frac{dy}{dx} = e^y\left(e^x + x^2 e^{x^3}\right) \quad \Rightarrow \int e^{-y}dy = \int\left(e^x + x^2 e^{x^3}\right)dx \quad \Rightarrow -e^{-y} = e^x + (1/3)e^{x^3} + C$

Example 5: Solve $(x+y)(dx - dy) = dx + dy$

Soln: Re-writting the given equation, we get $(x+y-1)dx = (x+y+1)dy \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1}$

Let $x + y = v \Rightarrow \frac{dy}{dx} = (dv/dx) - 1$

Therfore the given equation becomes,

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1} \quad \Rightarrow \frac{dv}{dx} = \frac{2v}{v+1} \quad \Rightarrow \int 2dx = \int\left(1 + \frac{1}{v}\right)dv$$

$$\Rightarrow 2x + c = v + \ln v \quad \Rightarrow -y + c = \ln(x+y)$$

Example 6: Solve $\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$

Soln: The given equation may be re-written as $\frac{dy}{dx} = \frac{2(2x+3y)+5}{(2x+3y)+4}$

Taking $2x + 3y = v \Rightarrow 2 + 3\frac{dy}{dx} = \frac{dv}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

Therefore, the given equation becomes,

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{2v+5}{v+4} \quad \Rightarrow \frac{dv}{dx} = \frac{3(2v+5)}{v+4} + 2 = \frac{8v+23}{v+4}$$

$$\Rightarrow \frac{dx}{dv}=\frac{v+4}{8v+23}=\frac{\frac{1}{8}(8v+23)+\left(4-\frac{23}{8}\right)}{8v+23}=\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right] \Rightarrow \int dx=\int\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right]dv$$

$$\Rightarrow x+c=\left(\frac{v}{8}\right)+\left(\frac{9}{64}\right)\log(16x+24y+23) \quad \Rightarrow 3y-6x+\left(\frac{9}{8}\right)\log(16x+24y+23)=8c$$

$$\Rightarrow y-2x+\left(\frac{3}{8}\right)\log(16x+24y+23)=c'$$ where $c'(=8c/3)$ is an arbitrary constant.

Example 7: Solution of the differential equation $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$ is

(a) $x+c=\log\left[1+\tan\left(\frac{x+y}{2}\right)\right]$ (b) $x+c=\log\left[1-\tan\left(\frac{x+y}{2}\right)\right]$

(c) $x+c=\log\left[1+\cot\left(\frac{x+y}{2}\right)\right]$ (d) $x+c=\log\left[1-\cot\left(\frac{x+y}{2}\right)\right]$

Soln: Put $x+y=z \Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$

Hence, given differential equation can be written as,

$$\frac{dz}{dx}-1=\sin z+\cos z \qquad \Rightarrow \int\frac{dz}{1+\sin z+\cos z}=\int dx$$

$$\Rightarrow \int\frac{dz}{1+\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}+\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}}=x+c$$

$$\Rightarrow \int\frac{\sec^2\frac{z}{2}\,dz}{1+\tan^2\frac{z}{2}+2\tan\frac{z}{2}+1-\tan^2\frac{z}{2}}=x+c \quad \Rightarrow \int\frac{\sec^2\frac{z}{2}\,dz}{1+2\tan\frac{z}{2}}=x+c$$

Put $1+2\tan\frac{z}{2}=t \Rightarrow 2\times\sec^2\frac{z}{2}\frac{dz}{2}=dt$

$$\therefore \int\frac{dt}{t}=x+c \Rightarrow \log(t)=x+c \Rightarrow x+c=\log\left[1+2\tan\left(\frac{x+y}{2}\right)\right].$$

Correct option is (a).

Example 8: Which of the following statement(s) about the solution $y(x)$ of the differential equation $\frac{1}{x^2-3}\frac{dy}{dz}=1$ is correct (with $y(0)=0$)

(a) $y(x)$ is given by $\frac{x^3}{3}-3x$