# Limits and Continuity

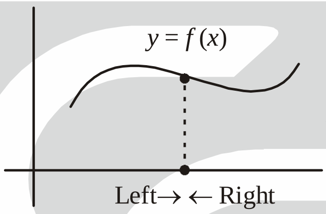
Chapter 2

**Limits:-** Recall that in the case of limit of functions of one variable, we say $\lim_{x \to x_0} f(x)$ exist if both $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist and equal. Here $x \to x_0^+$ and $x \to x_0^-$ reflecting the fact that there are only two directions from which *x* can approach to $x_0$, the right of *x* or left of *x*.

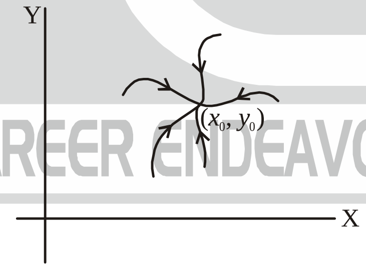
$$\underset{\text{approaching from left}}{x \to x_0^-} \quad \underset{\text{approching from right}}{x \to x_0^+}$$

$$x \to x_0 \leftarrow x$$

So limit is said to be exist if value of limit (if exist) along both possible path should be same.




But for a function of two variables, there are infinitely many directions in the 2-dimensional space along which (*x*, *y*) can approach to $(x_0, y_0)$, because there are infinite number of possible curves along which one point can approach another.




So we say that $\lim_{(x,y) \to (x_0 y_0)}$ exist, if limit exists along all the possible paths and has same value.

**Limits Along Curves:**

If *C* is a smooth parameteric curve in 2-dimensional space that is represented by the equations $x = x(t)$, $y = y(t)$, then

$$\lim_{\substack{(x,y) \to (x_0, y_0) \\ \text{along C}}} f(x, y) = \lim_{t \to t_0} f(x(t), y(t)) \text{ where } t_0 \text{ is parametric value for } (x_0, y_0)$$

**Example**

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

Find the limit of $f(x, y)$ at origin through the following curves?

(a) *x*-axis (b) *y*-axis (c) the line $y = mx$ (d) the parabola $y = x^2$

**Soln.** (a) The $x$-axis has parametric equations $x = t,\ y = 0$, so $\lim\limits_{\substack{(x,y)\to(0,0)\\ (\text{along } y=0)}} f(x, y) = \lim\limits_{t\to 0} f(t, 0) = \lim\limits_{t\to 0}\left(-\dfrac{0}{t^2}\right) = 0$.

(b) Similarly, do it as exercise.

(c) The line $y = mx$ has parametric equations $x = t,\ y = mt$, with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=mx)}} f(x, y) = \lim_{t\to 0} f(t, mt) = \lim_{t\to 0}\left(-\frac{m\cancel{t^2}}{(1+m^2)\cancel{t^2}}\right) = \lim_{t\to 0}\frac{-m}{1+m^2} = \frac{-m}{1+m^2}$$

i.e. the limit along $y = mx$ depends upon $m$. Consequently, it will be different along different lines through origin.

(d) The parabola $y = x^2$ has parametric equations $x = t,\ y = t^2$ with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

**Two-Paths Test of Limits:**

**Note that:** $\lim\limits_{(x,y)\to(x_0 y_0)} f(x, y)$ exist only when it is same along all the possible path along which $(x, y)$ approaches to $(x_0, y_0)$.

Therefore, if we can find out two different smooth curve (path) along which $(x, y)$ approaches $(x_0, y_0)$ but have different value of limit along them, we say function $f(x, y)$ do not have limit when $(x, y)$ approaches $(x_0, y_0)$.

**Example**

Find $\lim\limits_{(x,y)\to(0,0)} -\dfrac{xy}{x^2+y^2}$

**Soln.** Along curve $x = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0$$

Along curve $y = x$,

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} -\frac{xy}{x^2+y^2} = \lim_{x\to 0} -\frac{x^2}{x^2+x^2} = -\frac{1}{2}$$

Since, along two different curves, this limit has different values, the limit does not exist.

**Example**

Let

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}; & \text{if } x^4+y^2 \neq 0 \\ 0; & \text{if } x = y = 0 \end{cases}$$

show that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

**Soln.** If we approach (0, 0) along line $y = mx$, then $f(x, y) = f(x, mx) = \dfrac{mx^3}{x^4+m^2x^2} = \dfrac{mx}{x^2+m^2} \to 0$ as $x \to 0$.

But if we approach (0, 0) along curve $y = mx^2$, then $f(x, y) = f(x, mx^2) = \dfrac{mx^4}{x^4+m^2x^4} \to \dfrac{m}{1+m^2}$ which is different for different values of $m$. Hence, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

**Example**

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$ does not exist.

**Soln.** Along $x = my^2$, $\lim\limits_{y\to 0} \dfrac{2my^4}{y^4(m^2+1)} = \dfrac{2m}{m^2+1}$ which is different for different values of $m$. Hence, $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$ does not exist.

**General Definition of Limit:**

Given a function $f:\mathbb{R}^2 \to \mathbb{R}$. We say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$, whenever

(i) Every neighbourhood of the point $(x_0, y_0)$ contains a point of the domain of *f* different from $(x_0, y_0)$ and

(ii) For every $\in > 0$, there exists $\delta > 0$ such that if $(x, y)$ is in the domain and satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ (circular disk) then $|f(x,y) - L| < \in$

**Remark: (1)** By a neighbourhood of a point $(x_0, y_0)$, which means

(A) An open disc of centred at a point $(x_0, y_0)$ of radius *r*, that is

$$D_r(x_0,y_0) = \left\{(x,y)\in\mathbb{R}^2 : \sqrt{(x-x_0)^2+(y-y_0)^2} < r\right\}$$

OR

(B) A rectangular disc as $D_r(x_0,y_0) = \left\{(x,y)\in\mathbb{R}^2 : |x-x_0| + |y-y_0| < r\right\}$

(in definition we may have either circular disc or rectangular disc)

**(2)** The condition (i) is included because we donot want to consider limits for isolated points of the domain as in that case there is no “limiting process”. and the condition (ii) implies that as the distance between $(x, y)$ and $(x_0, y_0)$ tends to zero, the distance between $f(x, y)$ and *L* tends to zero.

**Example:** Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Let $x = r\cos\theta,\ y = r\sin\theta,$ then we have

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{r^2\cos\theta\sin\theta}{r}\right| = |r\cos\theta\sin\theta| \le r$$

$$= \sqrt{x^2+y^2} < \in$$

if $x^2 < \dfrac{\in^2}{2}, y^2 < \dfrac{\in^2}{2}$ OR if $|x| < \dfrac{\in}{\sqrt{2}}, |y| < \dfrac{\in}{\sqrt{2}}$. Then we have to choose $\in = \delta$,

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \in \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

**Example: (2)** $\lim\limits_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2}=0$

**Soln.** Let $x=r\cos\theta,\ y=r\sin\theta$, then

$$\left|f(x,y)-0\right|=\left|xy\frac{x^2-y^2}{x^2+y^2}-0\right|=\left|r^2\sin\theta\cos\theta\cdot\frac{r^2\cos 2\theta}{r^2}\right|$$

$$=\left|r^2\sin\theta\cos\theta\cos 2\theta\right|=\left|\frac{r^2}{4}\sin 4\theta\right|\le\frac{r^2}{4}$$

$$\le\frac{r^2}{4}=\frac{x^2+y^2}{4}<\in \text{ if } \sqrt{x^2+y^2}<4\sqrt{\in}=\delta(\text{let})$$

So for every $\in>0, \exists\ \delta=2\sqrt{\in}>0$ such that $\left|f(x,y)-0\right|<\in$ whenever $0<\sqrt{x^2+y^2}<\delta$

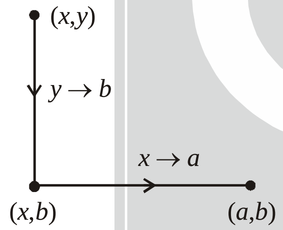
hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$

**Repeated Limits:**

We discussed about limits where $(x,y)\to(x_0,y_0)$ along some path $x$ and $y$ tends to $x_0$ and $y_0$ respectively. Now we discuss the concept of repeated limits.

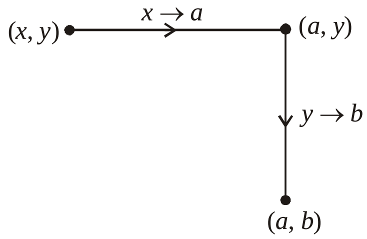
The repeated limit can be written as

$$\lim_{x\to a}\lim_{y\to b} f(x,y)=\lambda_1$$




If $\lim\limits_{y\to b} f(x,y)$ exist, then it is a function of $x$, say $\phi(x)$. if then the limit, $\lim\limits_{x\to a}\phi(x)$ exists and is equal to $\lambda_1$. Then $\lambda$ is a repeated limit of $f$ when first $y\to b$ and then $x\to a$.

Similarly $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)=\lambda_2$




If $\lim\limits_{x\to a} f(x,y)$ exist, then it is a function of $y$, say $\phi(y)$. If the limit $\lim\limits_{y\to b}\phi(y)$ exists and is equal to $\lambda_2$.

Then $\lambda_2$ is repeated limit of $f$ when first $x\to a$ and then $y\to b$.

Note that repeated limits may or may not be equal.

**Example: (1)** Find repeated limits of the function

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}; & (x,y)\ne(0,0)\\ 0 \quad ; & (x,y)=(0,0)\end{cases}$$

**Soln.** $$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)=\lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy}{x^2+y^2}\right)=\lim_{x\to 0}(0)=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)=\lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right)=\lim_{y\to 0}(0)=0$$

Here both repeated limit exists and are equal.

**Example: (2)** Find repeated limits of the function

$$f(x,y)=\begin{cases}\dfrac{x-y}{x+y} & ;\quad \text{if } x+y=0\\ 0 & ;\quad \text{if } x+y\neq 0\end{cases}$$

**Soln.** $$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)=\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right)=\lim_{x\to 0}\left(\frac{x}{x}\right)=\lim_{x\to 0}1=1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)=\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right)=\lim_{y\to 0}\left(\frac{-y}{y}\right)=\lim_{y\to 0}(-1)=-1$$

Here both repeated limit exists and are not equal.

**Remark:**

(1) If the repeated limits are not equal, then simultaneous limit cannot exist.

(2) If the simultaneous limit exists then the repeated limits if they exist are necessarily equal but the converse need not be true.

**Example: (3)** Find $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2xy}{x^2+y^2}$

**Soln.** Let $y=mx$, then we have

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,x)\to(0,0)}\frac{2xy}{x^2+y^2}=\lim_{x\to 0}\frac{2x\cdot mx}{x^2+m^2x^2}=\lim_{x\to 0}\frac{2mx^2}{(1+m^2)x^2}=\frac{2m}{1+m^2}$$

depends on the choice of slope $m$.

Hence limit do not exist.



Note that for $f(x,y)$ both repeated limit exists at (0, 0) and are equal.

**Example: (4)** Show that limit of the function

$$f(x,y)=\begin{cases}x\sin\left(\dfrac{1}{y}\right)+y\sin\left(\dfrac{1}{x}\right); & xy\neq 0\\ 0 \quad ; & xy=0\end{cases}$$

at (0, 0) is 0.

**Soln.** Consider

$$|f(x,y)-L|=\left|\left(x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right)-0\right|=\left|x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right|$$

$$\leq\left|x\sin\left(\frac{1}{y}\right)\right|+\left|y\sin\left(\frac{1}{x}\right)\right|\leq|x|+|y|$$

Let $\in>0$ be given and consider $\in=\delta$, we have $|f(x,y)-L|<\in$ whenever $0<|x-0|+|y-0|<\delta$