Chapter 2

First Order and First Degree-ODE

GENERAL FROM OF FIRST ORDER & FIRST DEGREE ORDINARY DIFFERENTIAL EQUATION

$$\frac{dy}{dx} = F(x, y) \text{ or } M(x, y)dx + N(x, y)dy = 0$$

METHODS OF SOLVING A DIFFERENTIAL EQUATION OF FIRST ORDER & FIRST DEGREE

Given an ordinary differential equation of I[st] order and I[st] degree $\frac{dy}{dx} = f(x, y)$

or $M(x, y)dx + N(x, y)dy = 0$ ...(i)

A differential equation of such type is not always solvable. However they can be solved when they belong to any of the type discussed in present articles.

Case I. **When $f(x, y)$ is function of only x or only y**

If the equation is of the form $\frac{dy}{dx} = f(x)$ ...(i)

and $\frac{dy}{dx} = f(y)$ ...(ii)

The equation of type (i) will reduce to $y = \int f(x)dx + C$. Whereas the equation of type (ii) will reduce to

$\frac{dy}{f(y)} = dx$ that can be solved as $\int \frac{dy}{f(y)} = x + C$; where C is parameter.

Case II. Variable Separable Form

The differential equation of the form $\frac{dy}{dx} = f(x) \bullet g(y)$ or $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ is called variable separable form because when expressed in form $M\,dx + N\,dy = 0$, the coefficients of dx is a function of x and dy is a function of y that means variables are separated by little rearrangement in the differential equation thus called as variable separable form.

So, the general form of such equation is $N(y)dy = M(x)dx$ which can be solved by integrating both sides i.e., $\int M(x)dx = \int N(y)dy$ as described as below for $\frac{dy}{dx} = \frac{f(x)}{g(y)}$.

Step I. Rearrange the expression to express it in the form $f(x)dx = g(y)dy$.

Step II. Integrating both sides we get $\int f(x)dx = \int g(y)dy$. Say $F(x)$, $G(y)$ be some anti-derivatives of $f(x)$ and $g(y)$ respectively we get, $G(y) = F(x) + C$.

Step III. Solving the equation, $G(y) = F(x) + C$ for y we express the general solution as $y = H(x, C)$.

Example-1

Solve the differential equation: $\frac{dy}{dx} = x.$

Soln. The given differential equation is : $dy = xdx$.

$\Rightarrow \int dy = \int xdx$

$$\Rightarrow y = \frac{x^2}{2} + C \quad \text{...(i)}$$

where C is an arbitrary constant.

Note : (i) is the general solution of the given differential equation.

Example-2

Solve the differential equation: $\frac{dy}{dx} = x - 1$ if $y = 0$ for $x = 1$.

Soln. The given differential equation is : $dy = (x - 1)\, dx$

$\Rightarrow \int dy = \int (x-1)\, dx$

$\Rightarrow y = \frac{x^2}{2} - x + C$ (General solution)

This is the general solution.

We can find value of C using $y = 0$ for $x = 1$.

$\Rightarrow 0 = \frac{1}{2} - 1 + C \quad \Rightarrow C = \frac{1}{2}$

$\Rightarrow y = \frac{x^2}{2} - x + \frac{1}{2}$ is the particular solution.

Example-3

Solve the differential equation: $(1+x)\,y\,dx + (1-y)\,x\,dy = 0$.

Soln. Separate the term of x and y get: $(1+x)ydx = -(1-y)xdy$

$\Rightarrow \frac{1+x}{x}dx = \frac{y-1}{y}dy$

$\Rightarrow \int \left(\frac{1+x}{x}\right)dx = \int \frac{y-1}{y}dy$

$\Rightarrow \log|x| + x = y - \log|y| + C$

$\Rightarrow \log|xy| + x - y = C$ is the general solution.

Example-4

Solve the differential equation: $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$.

Soln. The given differential equation: $xy^2 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$.

$$\Rightarrow xy^2 \frac{dy}{dx} = (1-x^2)(1+y^2) \Rightarrow \frac{y^2 dy}{1+y^2} = \frac{(1-x^2)\,dx}{x}$$

$$\Rightarrow \int \frac{y^2}{1+y^2} dy = \int \left(\frac{1}{x} - x\right) dx$$

$\Rightarrow y - \tan^{-1} y = \log|x| - \frac{x^2}{2} + C$ is the general solution of the given differential equation.

Example **-5**

Solve $\frac{d^2y}{dx^2} = x + \sin x$ if $y = 0$ and $\frac{dy}{dx} = -1$ for $x = 0$.

Soln. The given differential equation is:

$$\frac{d^2y}{dx^2} = x + \sin x \qquad ...(i)$$

It is an order 2 differential equation. But it can be easily reduced to order 1 differential equation by integrating both sides.

On Integrating both sides of equation (i), we get:

$$\frac{dy}{dx} = \int (x + \sin x)\,dx \Rightarrow \frac{dy}{dx} = \frac{x^2}{2} - \cos x + C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow dy = \left(\frac{x^2}{2} - \cos x + C_1\right) dx$$

$$\Rightarrow \int dy = \int \left(\frac{x^2}{2} - \cos x + C_1\right) dx \Rightarrow y = \frac{x^3}{6} - \sin x + C_1 x + C_2 \qquad ...(ii)$$

This is the general solution. For particular solution, we have to find C_1 and C_2.

Concept Application Exercise

1. Solve $\frac{dy}{dx}\sqrt{1 + x + y} = x + y - 1$.

2. Solve $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

3. Solve $ye^{x/y} dx = \left(xe^{x/y} + y^2\right) dy \, (y \neq 0)$.

4. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

5. $(x - y)(dx + dy) = dx - dy$, given that $y = -1$, where $x = 0$.

6. $\frac{dy}{dx} + y\, f'(x) = f(x)\, f'(x)$, where $f(x)$ is a given integrable function of x.

2. EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE FORM

Type A: $\frac{dy}{dx} = f(ax+by+c)$; where $b \neq 0$

Algorithm: The differential equations are of the form $\frac{dy}{dx} = f(ax+by+c)$ are reducible to variable separable form by substituting $ax+by+c=t$. The substitution reduces the differential equation to, $a + b\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{1}{b}\left(\frac{dt}{dx}-a\right) = f(t) \quad \Rightarrow \frac{dt}{dx} = bf(t)+a \quad \Rightarrow \int \frac{dt}{bf(t)+a} = \int dx$$

which can be solved as it is clearly in variable separable form.

Example-6

Solve the differential equation: $\frac{dy}{dx} - x\tan(y-x) = 1.$

Soln. The given differential equation is: $\frac{dy}{dx} - x\tan(y-x) = 1$

Put $z = y - x$

$$\Rightarrow \frac{dz}{dx} = \frac{dy}{dx} - 1 \quad \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} + 1$$

$\Rightarrow$ The given equation becomes :

$$\left(\frac{dz}{dx}+1\right) - x\tan z = 1$$

$$\Rightarrow \frac{dz}{dx} = x\tan z \Rightarrow \int \cot z dz = \int x dx$$

$$\Rightarrow \log|\sin z| = \frac{x^2}{2} + C$$

$$\Rightarrow |\sin(y-x)| = e^{x^2/2}.e^C$$

$$\Rightarrow \sin(y-x) = ke^{x^2/2},$$

where k is an arbitrary constant.

Type B: Equation of type: Any equation of the form: $R(x^2+y^2, x\,dx + y\,dy, x\,dy - y\,dx) = 0$.

Some times transformation to the polar co-ordinates facilities separation of variables.

Substitute: $x = r\cos\theta$ and $y = r\sin\theta$

$$\Rightarrow x^2 + y^2 = r^2 \quad ...(1)$$

and $\frac{y}{x} = \tan\theta$ ...(2)

Differentiating (1) w.r.t. any variable we get

$$x\,dx + y\,dy = r\,dr \quad ...(3)$$

Differenting (ii) w.r.t. x.

$$\Rightarrow \frac{\frac{x\,dy}{dx} - y}{x^2} = \sec^2\theta\frac{d\theta}{dx}$$

$$\Rightarrow x\,dy - y\,dx = x^2\sec^2\theta\,d\theta = r^2 d\theta \quad ...(4)$$

Now the equation reduces to $R(r^2, rdr, r^2 d\theta) = 0$.

Type D: Equation of type: Any equation of the form: $R(x^2 - y^2, x\,dx - y\,dy, x\,dy - y\,dx) = 0$.

Substitute: $x = r\sec\theta$ and $y = r\tan\theta$

$\Rightarrow \quad x^2 - y^2 = r^2$ ...(1)

and $\dfrac{y}{x} = \sin\theta$ ...(2)

Differentiate (1) w.r.t. any variable we get

$x\,dx - y\,dy = r\,dr$ ...(3)

Differentiate (2) w.r.t. x, we get $\dfrac{x\,dy - y\,dx}{x^2} = \cos\theta\,d\theta$

$\Rightarrow \quad x\,dy - y\,dx = r^2 \sec\theta\,d\theta$ ...(4)

Now the equation gets reduced to $R(r^2, rdr, r^2 \sec\theta\,d\theta) = 0$.

Type E: Equation of type: In an equation of the form: $yf_1(xy)dx + xf_2(xy)dy = 0$, the variable can be separated by the substitution $xy = v$.

Proof: $xy = v$

$\Rightarrow \quad y = \dfrac{v}{x} \qquad \Rightarrow \quad dy = \dfrac{xdv - vdx}{x^2}$,

$\Rightarrow \quad xdy = dv - \dfrac{v}{x}dx$ ...(1)

Using (i), $xy = v$ and $y = \dfrac{v}{x}$ in given differential equation $yf_1(xy)dx + xf_2(xy)dy = 0$, we get

$$\frac{v}{x} f_1(v)dx + f_2(v)\left\{dv - \frac{v}{x}dx\right\} = 0$$

$$\Rightarrow \quad \frac{f_2(v)dv}{v\{f_1(v) - f_2(v)\}} + \frac{dx}{x} = 0$$

Thus the variable are separated.

3. HOMOGENEOUS DIFFERENTIAL EQUATION

Homogeneous Function

$f(x, y)$ is said to be homogeneous expression of its variable of degree n, iff it can expressed as

$f(x, y) = x^n \phi\left(\dfrac{y}{x}\right)$ or $y^n \psi\left(\dfrac{x}{y}\right)$ it satisfies the identity $f(tx, ty) = t^n f(x, y)$.

For instance $f(x, y) = x^3 - 3xy^2 + y^3$ is homogeneous expression of degree 3 as

$f(tx, ty) = t^3x^3 - 3(tx)(ty) + 2(ty)^3 = t^3(x^3 - 3xy^2 + 2y^3) = t^3 f(x, y)$

eg.

(i) $f(x, y) = x^3 + x^2y + y^3 + xy^2 = x^3\left(1 + \dfrac{y}{x} + \dfrac{y^2}{x^2} + \dfrac{y^3}{x^3}\right)$ is homogeneous expression of degree 3.

(ii) $f(x, y) = \tan^{-1}\left(\dfrac{xy}{x^2 + y^2}\right) = x^\circ \tan^{-1}\left(\dfrac{y/x}{1 + y^2/x^2}\right)$ is homogeneous expression of degree 0.

Homogeneous Differential Equation