Method of Finding Solution:

Note that if we have a method of solving Quasi Linear Partial differential equations, then same method of is applicable for Semi Linear partial differential equation and linear partial differential equation. for a non-linear partial differential equation, we have different methods of finding solution. First we disscuss the method of solving Quasi linear partial differential equation.

Solution of Quasi Linear Equations of First Order:

Consider a first order quasi-linear partial differential equation as $P(x,y,u)u_x + Q(x,y,u)u_y - R(x,y,u) = 0$

We assume that the possible solution is of the form $u = u(x,y)$ or in an implicit form

$f(x,y,u) \equiv u(x,y) - u = 0$ represents a possible solution surface in *xyu*-space. At any point (x,y,u) of solution surface, the gradient vector $\nabla f = (f_x, f_y, f_u) = (u_x, u_y, -1)$ is normal to the solution surface.

Note that $Pu_x + Qu_y - R = (P,Q,R)\cdot(u_x,u_y,-1) = 0$

This shows that the vector (P,Q,R) must be a tangent vector of the integral surface at the point (*x, y, u*) and hence it determines a direction field called the characteristic direction.

In other words, $f(x,y,u) = u(x,y) - u = 0$ as a surface in the *xyu*-space is a solution iff the direction vector field (*P, Q, R*) lies in the tangent plane of the integral surface $f(x,y,u) = 0$ at each point (*x, y, u*).

Let $x = x(t), y = y(t)$ and $u = u(t)$, then the tangent vector $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right)$ must equal to (P,Q,R) So the characteristic equations are given as $\frac{dx}{dt} = P,\ \frac{dy}{dt} = Q, \frac{du}{dt} = R$ or $\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$

General Solution: The General solution of a first order quasi-linear partial differential equation $P(x,y,u)u_x + Q(x,y,u)u_y = R(x,y,u)$ is $F(\phi,\psi) = 0$ where F is arbitrary function of $\phi(x,y,u)$ and $\psi(x,y,u)$ and $\phi = c_1$ and $\psi = c_2$ are solution curve of characteristic equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$.

Remark: (i) we also write general solution as $c_2 = F(C_1)\ or\ C_1 = F(C_2)$.

(ii) Sometimes it is not possible to find two linearly independent solution using characteristic equations so we use method of lagrange, which state that we can write characteristic equation as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} = \frac{\ell dx + mdy + ndu}{\ell P + mQ + nR}$$ and solve any combination of these equaion.

(iii) If there exist some value of *l,m* and *n* such that *lP* + *mQ* + *nR* = 0, there we simple integrate the mumerator to get characteristic solution.

Ex.1: Solve the PDE $yzp + xzq = xyz$

Soln. The characteristic equation is given by $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xyz}$

Using $\frac{dx}{yz} = \frac{dy}{xz}$ we have $\frac{dx}{y} = \frac{dy}{x}$ i.e. $xdx - ydy = 0$

$\Rightarrow\ x^2 - y^2 = c_1$

Also using $\frac{dy}{xz} = \frac{dz}{xyz}$ *i.e.* $ydy = dz$

$\Rightarrow\ z - \frac{y^2}{2} = c_2$

so general solution is given by $F\left(x^2 - y^2, z - \frac{y^2}{2}\right) = 0$

Ex.2: Consider PDE $x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = u$ be the characteristic curves for above equation are

(a) straight lines with slope 1 (b) straight lines with slope –1
(c) circles with centre (0, 0) (d) circle touching y-axis

Soln. The characteristic equation is given by $\frac{dx}{-y} = \frac{dy}{x} = \frac{du}{u}$

using $\frac{dx}{-y} = \frac{dy}{x}$ we have $x\,dx = -y\,dy$

$\Rightarrow x^2 + y^2 = C_1^2$ which represents the circle with centre (0, 0).

Hence correct option is (c).

Ex.3: The integral curve of the equation $p + 2q = 5z + \cosh(2x - y)$ are given by the intersection of the surface.

(a) $2x - y = C_1$; $(5z + \cosh(2x - y)]e^{-5x} = C_2$

(b) $x - 2y = C_3$; $(5z + \cosh(x - 2y)]e^{-5x} = C_4$

(c) $2x - y = C_5$; $[z + \cosh(2x - y)]e^{-x} = C_6$

(d) $x - 2y = C_7$; $[z + \cosh(2x - y)]e^{-x} = C_8$ **[CSIR-NET Dec.-2008]**

Soln. The characteristic equation is given by $\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z + \cosh(2x - y)}$

using $\frac{dx}{1} = \frac{dy}{2}$, we get $2x - y = C_1$

Now using $\frac{dx}{1} = \frac{dz}{5z + \cosh C_1}$, we get $x = \frac{1}{5}\log\left[5z + \cosh C_1\right] + \log C_2$

$\Rightarrow\ 5x = C_2\left[5z + \cosh(2x - y)\right]$

$\Rightarrow \left[5z + \cosh(2x - y)\right]e^{-5x} = C_2$

Hence correct option is (a).

Ex.4: The general integral of the partial differential equation $(y + zx)\,z_x - (x + yz)\,z_y = x^2 - y^2$ is

(a) $F(x^2 + y^2 + z^2,\ xy + z) = 0$ (b) $F(x^2 + y^2 - z^2,\ xy + z) = 0$

(c) $F(x^2 - y^2 - z^2,\ xy + z) = 0$ (d) $F(x^2 + y^2 + z^2,\ xy - z) = 0$

where F is an arbitrary function. **[GATE-2001]**

Soln. $(y+zx)\,p-(x+yz)\,q=x^2-y^2$

The characteristic equation is given by $\dfrac{dx}{y+zx}=\dfrac{dy}{-(x+yz)}=\dfrac{dz}{x^2-y^2}$

using Lagrange multiplire $\dfrac{y\,dx+x\,dy+dz}{y^2-x^2+x^2-y^2}=\dfrac{d\left(xy+z\right)}{0} \Rightarrow xy+z=C_1$

Also $\dfrac{xdx+ydy+zdz}{0} \Rightarrow x^2+y^2+z^2=C_2$

The general solution is given by $f(x^2+y^2+z^2,\ xy+z)=0$

Hence correct option is (a).

Ex.5: The general solution of $2\dfrac{\partial z}{\partial x}+3\dfrac{\partial z}{\partial y}=\cos(3x-2y)$ is **CSIR-NET June-2007]**

(a) $2z=x\cos x(3x-2y)+f(3y+2x)$

(b) $2z=x\sin(3x-2y)+f(3x+2y)$

(c) $2z=x\cos(3x-2y)+f(3x-2y)$

(d) $2z=x\sin(3x-2y)+f(3y-2x)$

Soln. Given that $2\dfrac{\partial z}{\partial x}+3\dfrac{\partial z}{\partial y}=\cos(3x-2y)$

Lagrange's Auxiliary equation is $\dfrac{dx}{2}=\dfrac{dy}{3}=\dfrac{dz}{\cos(3x-2y)}$

Taking first two fraction $\dfrac{dx}{2}=\dfrac{dy}{3}$, we get $3x-2y=C_1$ ... (i)

Taking first and last fractional and put $C_1=3x-2y$, we have $\dfrac{dx}{2}=\dfrac{dz}{\cos C_1}$

$\Rightarrow \cos C_1\,dx=2dz$ *i.e.* $x\cos C_1=2z+C_2$ *i.e.* $x\cos(3x-2y)=2z+C_2$ *i.e.* $2z=x\cos(3x-2y)-C_2$

Hence general solution is $C_2=f\left(C_1\right)$ i.e. $2z=x\cos(3x-2y)+f(3x-2y)$.

Hence correct option is (c).

Ex.6: The general integral of $x(y-z)\dfrac{\partial z}{\partial x}+y(z-x)\dfrac{\partial z}{\partial y}=z(x-y)$ is **[CSIR-NET June-2007]**

(a) $F(2x+3y+z,\ xyz)=0$

(b) $F(x+y+z,\ xyz)=0$

(c) $F(x-y+z,\ xyz)=0$

(d) $F(x+y-z,\ xyz)=0$

Soln. Given that $x(y-z)\dfrac{\partial z}{\partial x}+y(z-x)\dfrac{\partial z}{\partial y}=z(x-y)$

Lagrange's Auxiliary equation is $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}$

$$\frac{dx+dy+dz}{xy-xz+yz-xy+xz-yz}=\frac{d\left(x+y+z\right)}{0} \Rightarrow x+y+z=C_1. \quad \text{... (i)}$$

Take a multiplier yz, xz, xy respectively, $\dfrac{yz\,dx+xz\,dy+xy\,dz}{xy^2z-xyz^2+xyz^2-x^2yz+x^2yz-xy^2z}=\dfrac{d\left(xyz\right)}{0} \Rightarrow xyz=C_2$ ... (ii)

Hence general integral is $F(x+y+z,\ xyz)=0$.

Hence correct option is (b).