Now, we will study about intervals, neighbourhoods, interior points, open sets, adherent points, closed sets, limit points, some examples and important results about the relations between sets.

3.1 Interval :

Let $S \subseteq \mathbb{R}$ then S is said to be an interval if for any $x, y \in S$ with $x \le t \le y \; \forall\, t \Rightarrow t \in S$

Note:

1. Singleton set is also an interval.

2. If S is interval sup S = β, inf S = α, then $\alpha - \beta$ is called length of interval.

3. When neither of the sup and the inf are member of the interval then the interval is open and is denoted by (α, β).

4. If the sup and the inf are member of the interval then interval is called closed and is denoted by $[\alpha, \beta]$.

5. Every subset of interval of finite length is bounded.

6. Set of natural numbers is not an interval.

7. Similarly Q and Q^C are not intervals but their union and intersection both are interval.(Empty set is an interval).

3.2 Neighbourhood:

$S \subseteq \mathbb{R}, a \in \mathbb{R}$, S is said to be a neighbourhood of 'a' if $\exists$ an open interval I s.t. $a \in I \subseteq S$

3.3 Interior Point:

Let $S \subseteq R$ and $a \in \mathbb{R}$. Then 'a' is said to be an interior point of S if S is a nbd of 'a'. i.e., if $\exists \delta > 0$ s.t. $(a - \delta, a + \delta) \subset S$.

3.4 Open set:

$S \subseteq \mathbb{R}$, we say S is open if it is a neighbourhood of all of it's points (elements) i.e., S is open its every point is interior point.

Remark: A set fails to be open if any of it's element is not an interior point.

Examples

1. So, ϕ is open.

2. Every open interval is an open set, but not conversely.

3. Non-empty open set is uncountable.

4. Set of rational numbers is not an open set since, no point of this set is an interior point.

3.5 Adherent point:

We say 'a' is an adherent point of S if for every $\delta > 0$, $(a - \delta, a + \delta) \cap S \ne \phi$.

Remarks:

1. All the elements of a set are adherent points but not conversely.

2. A Non-member can be an adherent point. e.g. $\sqrt{2}$ is adherent point of Q as $(\sqrt{2} - \delta, \sqrt{2} + \delta) \cap Q \ne \phi$ i.e., every nbd of 'a' contains atleast one element of S.

3.6 Limit point:

Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ be any set. We say 'a' is a limit point of S, if $(a - \delta, a + \delta) \cap S - \{a\} \ne \phi$ for every $\delta > 0$. i.e., every nbd of 'a' contains one element of S other than 'a'.

Example:

1. Limit point of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is 0

2. For (a, b), each point of this set is a limit point also. 'a' and 'b' does not belong to the set but they are also limit points of (a, b).

3. In [2, 3], all limit points belong to the set.

4. For $(a,b) \cap Q$, the collection of limit points of this set is $[a, b]$

3.7 Closed set:

Let S be any subset of $\mathbb{R}$. We say S is closed in $\mathbb{R}$ if $\mathbb{R}$ –S is open.

Remark:

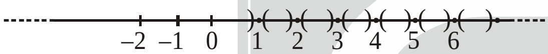
1. It is possible to have a set which is neither open nor closed e.g. Q.

2. It is possible to have a set which is both closed and open. e.g. Φ and $\mathbb{R}$.

Example:

1. Any closed interval is a closed set. $[a, b]$

2. Set of natural numbers is a closed set. Since it's compliment can be written as countable union of open sets. Hence it's compliment is an open set.



$$\mathbb{R} - \mathbb{N} = (-\infty, 1) \cup (1,2) \cup (2,3) \cup (3,4) \cup .....$$

3. Set of irrational numbers (Q^C) is not a closed set since it's compliment is not an open set.

4. The set [0, 1] $\cup$ [2, 3], which is not an interval, is closed.

5. The set $\mathbb{R}$ of real numbers is open as well as closed.

6. The set $\mathbb{Z}$ has no limit point, for a nbd $\left]m-\frac{1}{2}, m+\frac{1}{2}\right[$ of $m \in \mathbb{Z}$, contains no point of $\mathbb{Z}$ other than m.

3.8 Derived sets:

The set of all limit points of a set S is called the derived set of S and is denoted by S'.

Example: Obtain the derived sets:

1. $\{x : 0 \le x < 1\}$

2. $\{x : 0 < x < 1, x \in Q\}$

3. $\{1 + \frac{1}{n} : n \in \mathbb{N}\}$

4. $\{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$

5. Set of all irrational numbers

Answers.

1. [0, 1]

2. [0, 1]

3. {1}

4. $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is the set of all limit points.

5. Set of real numbers is the derived set of rational numbers.

Remarks:

1. A Neighbourhood is never a finite set.
2. A real number 'a' is a limit point of $S \subseteq \mathbb{R}$ iff every nbd of 'a' contains infinite elements of S.
3. A finite set can not have limit point.
4. Every limit point is an adherent point but not conversely.
   E.g. Finite sets have adherent points but no limit point.
5. Set of limit points is contained in the set of adherent points.
6. A Non-member adherent point is a limit point.

Definition: Isolated points:

Let S be any set of $\mathbb{R}$, 'a' $\in S$. Then 'a' is called an isolated point of S if it is not a limit point of S.

Table for some most common and important sets.

Set	Finite	Infinite	Open	Closed	Derived set	Interior set
$\{1/n\}$	×	✓	×	×	$\{0\}$	ϕ
$\{1, 2, 3\}$	✓	×	×	✓	ϕ	ϕ
$\mathbb{Q}$	×	✓	×	×	$\mathbb{R}$	ϕ
$\mathbb{N}$	×	✓	×	✓	ϕ	ϕ
$(a, b]$	×	✓	×	×	$[a, b]$	(a, b)
$[a, b]$	×	✓	×	✓	$[a, b]$	(a, b)
$[a, b] \cap \mathbb{Q}$	×	✓	×	×	$[a, b]$	ϕ
$\mathbb{R}$	×	✓	✓	✓	$\mathbb{R}$	$\mathbb{R}$
ϕ	✓	×	✓	✓	ϕ	ϕ

3.9 Bolzano-Weierstrass Theorem for sets:

Every infinite bounded set has a limit point.

Note: Bounded is not necessary for an infinite set S to have a limit point. The set $S = \{\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}, 5, ...\}$ is unbounded and infinite and has the limit point 'O'. The unbounded interval $]a, \infty[$ has infinitely many limit points.

Some examples:

1. If S and T are subsets of real numbers,
   (i) $S \subseteq T \Rightarrow S' \subseteq T'$ and (ii) $(S \cup T)' = S' \cup T'$
2. (i) If S, T are subsets of $\mathbb{R}$, then $(S \cap T)' \subseteq S' \cap T'$.
   (ii) Show that $(S \cap T)'$ and $S' \cap T'$ may not be equal.

Soln. Let S = (1, 2) and T = (2, 3), so,

$S \cap T = \phi \Rightarrow (S \cap T)' = \phi' = \phi$

Also, $S' = [1, 2], T' = [2, 3]$

$\therefore S' \cap T' = \{2\}$. Thus, $(S \cap T)' \neq S' \cap T'$

SOLVED EXAMPLES

1. For each $j = 1, 2, 3, ...,$ let A_j be a finite set containing atleast two distinct elements. Then

(a) $\bigcup_{j=1}^{\infty} A_j$ is a countable set

(b) $\bigcup_{n=1}^{\infty} \prod_{j=1}^{n} A_j$ is uncountable

(c) $\prod_{j=1}^{\infty} A_j$ is uncountable

(d) $\bigcup_{j=1}^{\infty} A_j$ is uncountable

[(MCQ) CSIR-NET/JRF : June-2012]

Soln. For each $j = 1, 2, 3, ...$

A_j = finite set containing atleast two distinct elements. Thus, every A_j is countable.

Hence, option (a) is correct and (d) is incorrect because, union of countable number of finite set is countable.

As 2^N is uncountable, so option (c) are uncountable.

Hence, option (a) and (c) are correct.

2. Let X denote the two point set $\{0, 1\}$ and write $X_j = \{0, 1\}$ for every $j = 1, 2, 3, ...$ Let $Y = \prod_{j=1}^{\infty} X_j$ which of the following is/are correct ?

(a) Y is countable set

(b) card Y = card $[0, 1]$

(c) $\bigcup_{n=1}^{\infty}\left(\prod_{j=1}^{n} X_j\right)$ is uncountable

(d) Y is uncountable

[(MCQ) CSIR-NET/JRF : June-2011]

Soln. $Y = \prod_{j=1}^{n} X_j$ is countable

$\therefore \bigcup_{n=1}^{\infty} \prod_{j=1}^{n} X_j$ is countable

($\because$ countable union of countable sets is countable)

Since $|Y| = 2^{|\mathbb{N}|} = |\mathbb{R}| = |[0,1]|$

Hence, option (b) and (d) are correct.

Some Important Results:

1. A set is closed iff its complement is open.

2. The intersection of an arbitrary family of closed sets is closed.

3. The union of two closed sets is a closed set.

Remark:

The result can be extended to the union of a finite number of closed sets but not for an arbitrary family. For example.

$S_n = [a + \frac{1}{n}, a + 2]$, for $n \in \mathbb{N}$ and $a \in \mathbb{R}$

Then $\bigcup_{n \in N} S_n =]a, a+2],$ which is not a closed set.

4. The derived set of a set is closed.

5. The derived set of a bounded set is bounded.

SOLVED EXAMPLES

1. Which one of the following set is not countable?

(a) $\mathbb{N}$, where $r \geq 1$ and $\mathbb{N}$ is the set of natural numbers

(b) $\{0,1\}^{\mathbb{N}}$, the set of all the sequences which takes values 0 and 1

(c) $\mathbb{Z}$, set of integers

(d) $\sqrt{2}\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers **[D.U. 2015]**

Soln. $\{0, 1\}^{\mathbb{N}}$, the set of all sequence which takes value 0 and 1. (a), (c) and (d) are countable by definition.

Hence, correct option is (b).

2. Let X be a countably infinite subset of $\mathbb{R}$ and A be a countably infinite subset of X. Then the set $X \backslash A = \{x \in X \mid x \notin A\}$

(a) is empty (b) is a finite set

(c) can be a countably infinite set (d) can be an uncountable set **[D.U. 2016]**

Soln. Let $X = \mathbb{Z} \subset \mathbb{R}$ and $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$ then $\mathbb{Z} \backslash \mathbb{N} \neq \phi$, not finite, $X \backslash A$ can't be uncountable as X is countable.

Although, it can be empty $(X = A)$ or finite $(\text{if } X = \mathbb{Z}^*, A = \mathbb{N})$

Hence, correct option is (c).

3. The subset $A = \{x \in \mathbb{Q} : x^2 < 4\}$ of $\mathbb{R}$ is

(a) bounded above but not bounded below

(b) bounded above and sup $A = 2$

(c) bounded above but does not have a supremum

(d) not bounded above **[D.U. 2016]**

Soln. $A = (-2, 2) \cap \mathbb{Q}$

bounded above and sup $A = 2$ as for every $\in > 0\ \exists u \in \mathbb{Q} \cap (-2, 2)$ such that $2 - \in < u$

Hence, correct option is (b).

4. The subset $A = \{x \in \mathbb{Q} : -1 < x < 0\} \cup \mathbb{N}$ of $\mathbb{R}$ is

(a) bounded, infinite set and has a limit point in $\mathbb{R}$

(b) unbounded, infinite set and has a limit point in $\mathbb{R}$

(c) unbounded, infinite set and does not have a limit point in $\mathbb{R}$

(d) bounded, infinite set and does not have a limit point in $\mathbb{R}$ **[D.U. 2016]**

Soln. Unbounded infinite set and has limit points in $\mathbb{R}$.

Hence, correct option is (b).

PRACTICE SET-1

[Bounded sets, Interior points, Limits points, Open and closed sets]

1. True / False
   (i) A subset of bounded set is bdd.
   (ii) Finite intersection of nbd need not be nbd.
   (iii) Open set may or may not be open interval.
   (iv) A^o is the largest open set contained in A. [where $A^o \to$ set of interior points of A]
   (v) For any set A; $(A^o)^o = A^o$
   (vi) The intersection of arbitrary family of closed sets need not be closed.
   (vii) Isolated points of a set are always countable.
   (viii) Every interior point of a set is a limit point and conversely.
   (ix) A is closed iff $\overline{A} = A$. [where $\overline{A}$, closure of A]
   (x) Adherent point may or may not be limit point.

Soln. (i) T, (ii) F, (iii) T, (iv) T, (v) T, (vi) F, (vii) T, (viii) F, (ix) T, (x) T.

2. One word answer type questions:
   (a) Find the intersection of sets $S_n = (-\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}$.
   (b) Obtain the derived set of $\{[0,8] \cap Q\} \cup \{1,2,3\} \cup (-3,0)$
   (c) What can you say about closedness of the set $\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, ...\}$
   (d) Find the supremum and the infimum of the above set.
   (e) Is the supremum or infimum lying in the derived set of the above set.
   (f) What is Closure of Q and interior of Q.

Soln. (a) {0}, (b) [–3, 8], (c) closed, (d) $\sup = \frac{3}{2}, \inf = \frac{-3}{2}$
(e) No (f) R and ϕ.

3. Multiple choice with single correct answer type question:
   (i) Let S be an infinite subset of $\mathbb{R}$ such that $S \cap Q = \phi$ which of the following is true .
   (a) S must have a limit point which belongs to Q.
   (b) S must have a limit point which belongs to $\mathbb{R}$/Q.
   (c) S cannot be a closed set in $\mathbb{R}$.
   (d) $\mathbb{R}$/S must have a limit point which belongs to S. Ans. (d)

   (ii) Consider the following subsets of $\mathbb{R}$:
   $$E = \left\{\frac{n}{n+1}, n \in \mathbb{N}\right\}, F = \left\{\frac{1}{1-x}, 0 \le x < 1\right\}$$
   then
   (a) Both E and F are closed. (b) E is closed and F is not closed.
   (c) E is not closed and F is closed. (d) Neither E nor F is closed. Ans. (c)

   (iii) $S = \left[(-1)^n\left(\frac{1}{4} - \frac{4}{n}\right), n \in \mathbb{N}\right]$ then

(a) sup $S=\frac{1}{4}$ and inf $S=\frac{1}{4}$
(b) sup $S=\frac{15}{4}$ and inf $S=-\frac{7}{4}$
(c) sup $S=\frac{15}{4}$ and inf $S=-\frac{1}{4}$
(d) None of these
Ans. (b)

(iv) Consider the statement

[1] $l\in\mathbb{R}$ is limit point of $A\cup B$ iff l is limit point of A or limit point of B.

[2] $l\in\mathbb{R}$ is limit point of $A\cap B$ iff l is limit point of A and limit point of B. Then

(a) 1 is correct 2 incorrect
(b) 2 correct 1 incorrect
(c) Both are correct
(d) Both are incorrect
Ans. (d)

(v) If $S=\{x:x^2<2$ and x is rational$\}$. Then derived set of S is

(a) $(-\sqrt{2},\sqrt{2})$
(b) $[-\sqrt{2},\sqrt{2}]$
(c) [–2, 2]
(d) (–2, 2)
Ans. (b)

(vi) Select the correct statement:

(a) Every bounded set is closed
(b) Every unbounded set is open
(c) Every closed set is countable
(d) Every non-empty open set is uncountable
Ans. (d)

(vii) S be subset of $\mathbb{R}$ and inf S = sup S, then

(a) S is empty
(b) S is singleton
(c) S finite but may not be singleton
(d) None of these
Ans. (c)

4. Multiple choice with one or more correct answer type questions:

(i) Choose the correct statement (s)

(a) Non-empty countable set cannot be open
(b) For a non-empty set uncountability is a necessary condition for the set to be open.
(c) Uncountability is sufficient condition for a set to be open.
(d) A closed set can be uncountable or countable.

(ii) Choose the correct statement (s)

(a) If S and T are two non-empty subset of $\mathbb{R}$ then sup ($S\cup T$) = max. {sup S, sup T}
(b) If S and T are two non-empty subset of $\mathbb{R}$ then inf ($S\cup T$) = min. {inf S, inf T}
(c) If A and B are two non-empty subset of $\mathbb{R}$ then sup ($A\cap B$) = min. {sup A, sup B}
(d) If A and B are two non-empty subset of $\mathbb{R}$ then sup ($A\cap B$) = max. {inf A, inf B}

(iii) Let $E\subset\mathbb{R}, E\neq\phi$. Let (1), (2) and (3) denote the following conditions: choose correct.

(1) E is infinite (2) E is bounded (3) E is closed

(a) 1 is necessary for E to have a limit point.
(b) 1 and 2 together are sufficient for E to have a limit point
(c) 1 and 3 together are sufficient for E to have a have a limit point
(d) 3 is sufficient for every limit point of E to belong to E

(iv) Let R be a set which is bounded below and S is the set of all its lower bounds. Then which of the following is incorrect?

(a) Both R and S have finite elements

(b) S has supremum and R has infimum

(c) Both S and R have infimum

(d) Both S and R have supremum

(v) Select the correct statement about the set $A = \{1, 1+\frac{1}{3}, 1+\frac{1}{3}+\frac{1}{3^2}, ..., 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+.....+\frac{1}{3^{n-1}}\}$

(a) The set A is bounded

(b) The inf A is 1

(c) The set A has its largest elements

(d) None of these

PRACTICE SET - 1

Answers key

4.(i) (a), (b), (d) **(ii). (a), (b) for (c), (d) counter axis (1, 2) and (2, 3)**

(iii).(a), (b) **(iv).(a), (c), (d)** **(v). (a), (b)**