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Gradient, Divergence & Curl

GRADIENT, DIVERGENCE AND CURL:

Del operator :-- Del operator, written as ∇, is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

This operator is also known as 'nabla'.

Gradient :-Let ϕ defines a differentiable scalar field, then gradient of ϕ written as $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Note :(i) $\nabla\phi$ defines a vector field.

(ii) $\nabla\phi = \sum i\frac{\partial\phi}{\partial x}$

Exp.1. If $\phi = 3x^2y$,then find gradiant of ϕ.

Soln. $\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y) = \hat{i}\left(\frac{\partial}{\partial x}(3x^2y)\right) + \hat{j}\frac{\partial}{\partial y}(3x^2y) + \hat{k}\frac{\partial}{\partial z}(3x^2y)$

$$= \hat{i}(6xy) + \hat{j}(3x^2) + \hat{k}(0) = 6xy\,\hat{i} + 3x^2\,\hat{j}$$

Result : If $\phi(x,y,z) = c$, then $\nabla\phi(x,y,z)$ is normal to the surface $\phi(x,y,z) = c$ at the point (x, y, z).

Ex.2. If $xyz = 1$. Then find the normal at point (1,1,1) to the given surface.

Soln. Given, $xyz = 1$

$\Rightarrow xyz - 1 = 0$

Let $\phi(x,y,z) = xyz - 1$

$$\therefore \nabla\phi = \nabla(xyz-1) = \frac{\partial(xyz-1)}{\partial x}\hat{i} + \frac{\partial(xyz-1)}{\partial y} + \frac{\partial(xyz-1)}{\partial z} = yz\hat{i} + xz\hat{j} + xy\hat{k} \Rightarrow \nabla\phi|_{(1,1,1)} = \hat{i} + \hat{j} + \hat{k}$$

Ex.3. Find the point at which the gradient of the function $f(x,y) = \ln\left(x + \frac{1}{y}\right)$ is equal to $\hat{i} - \frac{16}{9}\hat{j}$

Slon. grad $f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \frac{1}{x+\frac{1}{y}}\hat{i} - \frac{\frac{1}{y^2}}{x+\frac{1}{y}}\hat{j} = \frac{y}{xy+1}\hat{i} - \frac{1}{y(xy+1)}\hat{j} \Rightarrow \hat{i} - \frac{16}{9}\hat{j} = \frac{y}{xy+1}\hat{i} - \frac{1}{y(xy+1)}\hat{j}$$

By compairing,

$$\frac{y}{xy+1}=1 \text{ and } \frac{1}{y(xy+1)}=\frac{16}{9}$$

$$\Rightarrow \frac{1}{y^2}=\frac{16}{9} \Rightarrow y=\pm\frac{3}{4}$$

When, $y = 3/4$

$$y=1+xy \Rightarrow \frac{3}{4}=1+\frac{3}{4}x \Rightarrow x=-\frac{1}{3}$$

When, $y=\frac{-3}{4}$

$$xy+1=y \Rightarrow \frac{-3}{4}x+1=\frac{-3}{4} \Rightarrow x=\frac{7}{3}$$

Therefore points are $\left(\frac{7}{3},-\frac{3}{4}\right) \& \left(-\frac{1}{3},\frac{3}{4}\right)$

Ex.4: Find the points at which the modulus of the gradient of the function $f(x,y)=\left(x^2+y^2\right)^{3/2}$ is equal to 2.

Soln. We have,

$$f(x,y)=\left(x^2+y^2\right)^{3/2}$$

$$\Rightarrow \nabla f=\frac{\partial\left(x^2+y^2\right)^{3/2}}{\partial x}\hat{i}+\frac{\partial\left(x^2+y^2\right)^{3/2}}{\partial y}\hat{j}+\frac{\partial\left(x^2+y^2\right)^{3/2}}{\partial z}\hat{k}=3x\left(x^2+y^2\right)^{1/2}\hat{i}+3y\left(x^2+y^2\right)^{1/2}\hat{j}$$

Given, $|\nabla f|=2 \Rightarrow \sqrt{9x^2\left(x^2+y^2\right)+9y^2\left(x^2+y^2\right)}=2$

$$\Rightarrow \sqrt{9\left(x^2+y^2\right)^2}=2 \Rightarrow 3\left(x^2+y^2\right)=2 \Rightarrow x^2+y^2=2/3$$

Points on this circle.

Ex.5: If $f(u,v)=\phi(u,v), u=\psi(x,y,z), v=\xi(x,y,z)$

Show that,

$$\text{grad } f=\frac{\partial\phi}{\partial u}\text{grad } u+\frac{\partial\phi}{\partial v}\text{grad } v.$$

Soln. We have, $f(u,v)=\phi(u,v)$

$$\Rightarrow \frac{\partial f}{\partial x}=\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x},\ \frac{\partial f}{\partial y}=\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y} \text{ and } \frac{\partial f}{\partial z}=\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial z}$$

$$\therefore \text{ grad } f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}=\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}\hat{i}+\frac{\partial u}{\partial y}\hat{j}+\frac{\partial u}{\partial z}\hat{k}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}\hat{i}+\frac{\partial v}{\partial y}\hat{j}+\frac{\partial v}{\partial z}\hat{k}\right)$$

$$=\frac{\partial\phi}{\partial u}\text{grad } u+\frac{\partial\phi}{\partial v}\cdot\text{grad } v$$

Divergence: Let $\vec{V}$ deifines a differentiable vector field, Then divergence of $\vec{V}$, written $\nabla\cdot\vec{V}$ or div$\vec{V}$, is defined by

$$\nabla\cdot\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\bullet\left(V_1\hat{i}+V_2\hat{j}+V_3\hat{k}\right)=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}$$

Note : (i) $\nabla\cdot\vec{V}\neq\vec{V}\cdot\nabla$

(ii) $\nabla\cdot\vec{V}=\sum\hat{i}\cdot\frac{\partial}{\partial x}(\vec{V})$

Exp.6: If $\vec{a}=xy^2\,\vec{i}+2x^2yz\,\vec{j}-3yz^2\,\vec{k}$, then find the divergence of vector $\vec{a}$

Soln. $\nabla\cdot\vec{a}=\frac{\partial}{\partial x}\left(xy^2\right)+\frac{\partial}{\partial y}\left(2x^2yz\right)+\frac{\partial}{\partial z}\left(-3yz^2\right)$

$=y^2+2x^2z-6yz$

Note: $\nabla\cdot\vec{a}$ is a scalar function.

Curl : If $\vec{V}(x,y,z)$ is a differentiable vector field then the curl or rotation of $\vec{V}$, written $\nabla\times\vec{V}$, curl $\vec{V}$ or rotation $\vec{V}$ is defined by,

$$\nabla\times\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}\,\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(V_1\hat{i}+V_2\hat{j}+V_3\hat{k}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_1 & V_2 & V_3\end{vmatrix}$$

$$=\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)\hat{i}+\left(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right)\hat{j}+\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)\hat{k}$$

Note: (i) If $\nabla\times\vec{a}=0\Leftrightarrow\exists$ a scalar field ϕ such that $\vec{a}=\nabla\phi$.

(ii) $\nabla\times\vec{V}=\sum i\times\frac{\partial}{\partial x}\left(\vec{V}\right)$

Properties:

1. div $\left(\vec{a}\pm\vec{b}\right)=$ div $\vec{a}\pm$ div $\vec{b}$
2. curl $\left(\vec{a}\pm\vec{b}\right)=$ curl $\vec{a}\pm$ curl $\vec{b}$
3. div $\left(u\,\vec{a}\right)=u\,div\,\vec{a}+$ grad $u\cdot\vec{a}$, where u is a scalar point function or scalar field
4. curl $\left(u\,\vec{a}\right)=u$ curl $\vec{a}+$ (grad u)$\times\vec{a}$
5. div $\left(\vec{a}\times\vec{b}\right)=\vec{b}\cdot$ curl $\vec{a}-\vec{a}\cdot$ curl $\vec{b}$
6. curl $\left(\vec{a}\times\vec{b}\right)=\vec{a}$ div $\vec{b}-\vec{b}$ div $\vec{a}+\left(\vec{b}\cdot\nabla\right)\vec{a}-\left(\vec{a}\cdot\nabla\right)\vec{b}$
7. grad $\left(\vec{a}\cdot\vec{b}\right)=\vec{a}\times$ curl $\vec{b}+\vec{b}\times$ curl $\vec{a}+\left(\vec{a}\cdot\nabla\right)\vec{b}+\left(\vec{b}\cdot\nabla\right)\vec{a}$
8. $\nabla\times\left(\nabla\phi\right)=0$ i.e. the curl of gradient of ϕ is zero.

9. $\nabla\cdot(\nabla\times\vec{a})=0$ i.e. divergence of curl of $\vec{a}$ is zero.

10. $\nabla\cdot\nabla\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\nabla^2\phi$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is known as laplacian operator.

11. $\nabla\times(\nabla\times\vec{a})=\nabla(\nabla\cdot\vec{a})-\nabla^2\vec{a}$

12. If $\vec{V}$ denote velocity of fluid, then if div $\vec{V}=0\Rightarrow$ fluid is incompressible.

13. A vector field $\vec{V}$ is said to be **solenoidal** if div $\vec{V}=0$.

14. A vector field $\vec{V}$ is said to be **irrotational** field if curl $\vec{V}=0$. If curl $\vec{V}\neq 0$, then its is called a rotational field.

Prove Property 3:

$$div(u\vec{a})=\sum\hat{i}\cdot\frac{\partial}{\partial x}(u\vec{a})=\sum\hat{i}\cdot\left(u\frac{\partial\vec{a}}{\partial x}+\vec{a}\frac{\partial u}{\partial x}\right)=\sum u\left(i\cdot\frac{\partial\vec{a}}{\partial x}\right)+\sum\left(\frac{\partial u}{\partial x}i\cdot\vec{a}\right)=u\,\text{div}\vec{a}+\vec{a}\cdot\text{grad}\,u$$

Prove Property 4 :

$$\text{Curl}(u\vec{a})=\sum i\times\left(\frac{\partial}{\partial x}(u\vec{a})\right)=\sum i\times\left(\vec{a}\frac{\partial u}{\partial x}+u\frac{\partial\vec{a}}{\partial x}\right)=\sum\left(\frac{\partial u}{\partial x}i\right)\times\vec{a}+u\sum i\times\frac{\partial\vec{a}}{\partial x}$$

$$=\text{gradu}\times\vec{a}+u\,\text{curl}\,\vec{a}$$

Prove Property 5:

$div(\vec{a}\times\vec{b})=\vec{b}\cdot\text{curl}\vec{a}-\vec{a}\cdot\text{curl}\vec{b}$

Prove $div(\vec{a}\times\vec{b})=\sum i\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{b})=\sum i\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}+\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right)$

$$=\sum i\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right)+\sum i\cdot\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right)=\sum\left(i\times\frac{\partial\vec{a}}{\partial x}\right)\cdot\vec{b}+\sum\vec{a}\cdot\left(\frac{\partial\vec{b}}{\partial x}\times i\right)$$

$$=(\text{curl}\,\vec{a})\cdot\vec{b}+\vec{a}\cdot(-\text{curl}\,\vec{b})=\vec{b}\cdot\text{curl}\,\vec{a}-\vec{a}\cdot\text{curl}\,\vec{b}$$

Rest do yourself

Exp.7: If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$, then the value of $div(r^n\vec{r})$ is **(BHU - 2014)**

(a) 0 (b) nr^{n-1} (c) nr^n (d) $(n+3)r^n$

Soln. $div(r^n\vec{r})=\nabla\cdot(r^n\vec{r})$

$$=\frac{\partial}{\partial x}(r^nx)+\frac{\partial}{\partial y}(r^ny)+\frac{\partial}{\partial z}(r^nz)=\left(x\cdot nr^{n-1}\cdot\frac{\partial r}{\partial x}+r^n\right)+\left(y\cdot nr^{n-1}\cdot\frac{\partial r}{\partial y}+r^n\right)+\left(z\cdot nr^{n-1}\cdot\frac{\partial r}{\partial z}+r^n\right)$$

$$=xnr^{n-1}\frac{x}{r}+ynr^{n-1}\frac{y}{r}+znr^{n-1}\frac{z}{r}+3r^n=nx^2r^{n-2}+ny^2r^{n-2}+nz^2r^{n-2}+3r^n$$

$$=nr^{n-2}(x^2+y^2+z^2)+3r^n=nr^n+3r^n=(n+3)r^n$$

Exp.8: If $\vec{r}$ is the position vector of the point (x, y, z) w.r.t. origin, prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. Find $f(r)$ such that $\nabla^2 f(r) = 0$.

Soln. $\nabla^2 f(r) = \sum \frac{\partial^2}{\partial x^2} f(r) = \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right) = \sum \frac{\partial}{\partial x}\left(f'(r)\cdot \frac{x}{r}\right)$

$$= \sum \left[\frac{f'(r)}{r} + \frac{x}{r} f''(r)\cdot \frac{x}{r} + x f'(r)\cdot\left(-\frac{1}{r^2}\right)\cdot \frac{x}{r}\right] = \sum\left[\frac{f'(r)}{r} + \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r)\right]$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{1}{r} f'(r) = f''(r) + \frac{2}{r} f'(r)$$

Now, let us find $f(r)$ such that $\nabla^2 f(r) = 0$

$$\Rightarrow f''(r) + \frac{2}{r} f'(r) = 0 \quad \Rightarrow f''(r) = -\frac{2}{r} f'(r) \Rightarrow \frac{f''(r)}{f'(r)} = -\frac{2}{r}$$

Integrating w.r.t r,

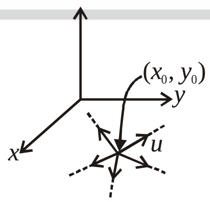
$$\Rightarrow \ln f'(r) = -2\ln r + \ln c \quad \Rightarrow f'(r)\cdot r^2 = c \Rightarrow f'(r) = \frac{c}{r^2}$$

Again integrating we get,

$$\boxed{f(r) = -\frac{c}{r} + d}$$

Directional Derivatives:

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in directions parallel to the x-axis and y-axis respectively. In this section we will discuss rates of change of $f(x, y)$ in other directions.



Definition: If $f(x, y)$ is a function of x and y, and if $\hat{u} = u_1\hat{i} + u_2\hat{j}$ is a unit vector, then the **directional derivative of *f* in the direction of *u*** at (x_0, y_0) is denoted by $D_{\vec{u}} f(x_0, y_0)$ and is defined by

$$D_{\vec{u}} f(x_0, y_0) = \frac{d}{ds}[f(x_0 + su_1, y_0 + su_2)]_{s=0}$$ provided the derivative exists

or $D_{\vec{u}} f(x_1, y_1) = \lim_{h\to 0} \frac{f(x_1 + u_1 h, x_2 + u_2 h) - f(x_1, x_2)}{h}$, provided the derivative exists