Chapter 3

WORK, ENERGY & POWER

Conservative force: A force is said to be conservative if curl of the force is zero.

i.e. if $\vec{\nabla}\times\vec{F}=0$ then, $\vec{F}$ is conservative force.

Work done by a conservative force is independent of path followed and work done along a closed path is zero. Gravitational and electrostatic forces are conservative. Friction, viscous forces are not conservative.

Potential energy: For a single particle, potential energy is defined as the work needed to be them to bring it to a particular point from a reference point. The reference point is usually taken at infinity. If a force field $\vec{F}$ brings the particle from infinity to a particular point, then the potential energy is defined as:

$$U=-\int\vec{F}\cdot d\vec{r} \text{ or } U=-\int F dx \quad (\text{For one dimension})$$

Spring Force: When a spring is either stretched or compressed such that its length changes by x, then restoring force or tension in the spring is $(-kx)$.

Spring potential energy: Particle attached to the spring is always attracted towards mean position. Therefore, $F=-kx$

$$\therefore U=-\int(-kx)dx=\frac{1}{2}kx^2$$

Gravitational potential energy for two particles:

We know that gravitational force between two particles of masses m_1 and m_2 at separation r is always attractive and is equal to

$$f=-\frac{Gm_1.m_2}{r^2} \qquad \therefore U=-\int\left(-\frac{Gm_1.m_2}{r^2}\right)dr=-\frac{Gm_1.m_2}{r}$$

Work: It is defined as $W=\int\vec{F}.\overrightarrow{dl}$

where $\overrightarrow{dl}$ is infinitesimal displacement of the object on which force is acting.

Power: It is defined as dot product of force and instantaneous velocity $P=\vec{F}.\vec{v}$

Average power is defined as total work done divided by total time $\langle P\rangle=\frac{W}{t}$

Kinetic Energy: For a particle of mass 'm' and speed v, kinetic energy is defined as $K=\frac{1}{2}mv^2$

In terms of momentum it is written as $K=\frac{p^2}{2m}$.

Work-energy theorem: It states that total work done by all forces is equal to change in kinetic energy of system.

$$W = K_{final} - K_{initial}$$

Conservation of mechanical energy: If only conservative forces are acting on a system then total mechanical energy of system remains conserved. Therefore

$$E_{initial} = E_{final}$$

or $\quad K_i + U_i = K_f + U_f \;\Rightarrow\; U_i - U_f = K_f - K_i$

Therefore, loss in potential energy = gain in kinetic energy (or vice versa).

SOLVED EXAMPLES

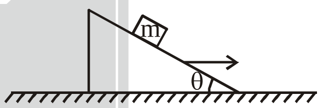
1. A particle moves from point (1, –1, 2) to (2, 3, 5) when a force $\vec{F} = 3\hat{i} + 5\hat{j} + 4\hat{k}$ acts on it. Calculate work done by the force.

Soln. The given force is constant. Therefore it is a conservative force. Therefore work done is independent of path followed.

$$W = \int \vec{F}.d\vec{l} = \int F_x.dx + \int F_y dy + \int F_z dz = \int_1^2 3dx + \int_{-1}^{3} 5dy + \int_2^5 5dz$$

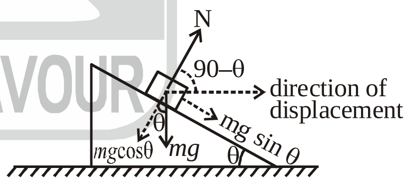
$$= 3(2-1) + 5(3+1) + 5(5-2) = 38 \text{ Joule}$$

2. A block of mass *m* is lying on a wedge as shown in figure. The block does not slide on the wedge while the wedge moves horizontally with uniform speed.
Calculate work done by normal reaction on the block during the time wedge moves by a distance S in horizontal direction.



Soln. From the figure shown we find that $N = mg\cos\theta$

and angle between N and direction of displacement is $90 - \theta$. Therefore work done by normal reaction is,



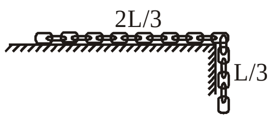
$$W = \int \vec{F}.d\vec{l} = \vec{N}.\int d\vec{l} = \vec{N}.\vec{S}$$

$$= N\,S\cos(90-\theta) = mg\cos\theta.S\sin\theta$$

$$= mg\,S\sin\theta.\cos\theta$$

3. Two third length of a thin metallic chain of mass per unit length λ is held on a smooth table and remaining hanging off from its edge. If the chain is released what will be its speed at the moment the chain completely slips off the table.

Soln. There is no dissipative force acting on the system.
Therefore total energy of the system remains conserved. Therefore we will use conservation of energy to find speed. Let us take potential energy to be zero at level of table top. Initially centre of mass of hanging part is L/6 below the table top and finally centre of mass of whole chain is L/2 below the table top. Therefore,



$$U_{initial} = -\left(\frac{\lambda L}{3}\right)g\left(\frac{L}{6}\right) = -\frac{\lambda g L^2}{18}, \quad U_{final} = -(\lambda L)g\left(\frac{L}{2}\right) = -\frac{\lambda g L^2}{2}$$

$$K_{initial} = 0, \quad K_{final} = \frac{1}{2}(\lambda L)v^2$$

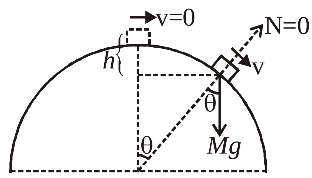
From conservation of energy $K_{initial} + U_{initial} = K_{final} + U_{final}$

$$0 + \left(-\frac{\lambda g L^2}{18}\right) = \frac{1}{2}(\lambda L)v^2 + \left(-\frac{\lambda g L^2}{2}\right) \quad \Rightarrow v = \sqrt{\frac{8gL}{9}}$$

4. A small object is released at the top of a smooth fixed sphere due to which it slides on the sphere for some time and then breaks off (leaves) the sphere. Calculate the angle with upward vertical at the position where object leaves the sphere.

Soln. Let the angle with upward vertical is θ. Therefore height descended by object from its initial position.

$h = R(1 - \cos\theta)$



from conservation of energy

loss in potential energy = gain in kinetic energy

$$mgR(1 - \cos\theta) = \frac{1}{2}mv^2$$

$\therefore v^2 = 2Rg(1 - \cos\theta)$ ... (a)

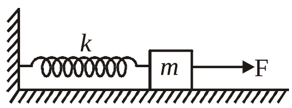
At the moment of leaving the sphere the object is still moving on circular path of radius R and the normal force becomes zero. Therefore,

Net force toward centre = $\dfrac{mv^2}{R}$, or $mg\cos\theta = \dfrac{mv^2}{R}$

or $v^2 = Rg\cos\theta$ ... (b)

from (a) and (b) we get $\cos\theta = \dfrac{2}{3}$ or $\theta = \cos^{-1}\dfrac{2}{3}$

5. A block of mass m attached to a light spring of spring constant k lies in on a smooth horizontal plane as shown in figure. A horizontal force F is applied to the block.



Calculate maximum speed of the block.

Also calculate maximum elongation in spring.

Soln. When force is applied, the block accelerates for some time then it decelerates. Therefore acceleration of block is initially positive, then it momentarily becomes zero and finally becomes negative till the block stops. Speed of block is maximum at the instant when its acceleration is zero.

$$a_x = 0 \Rightarrow F_x = 0 \qquad \therefore F - kx = 0 \Rightarrow x = \frac{F}{k}$$

Thus speed is maximum when elongation in spring is equal to $\dfrac{F}{k}$

To find maximum speed let us use work energy theorem.

Total work done = change in kinetic energy

$$W_{Spring} + W_{Force} = \frac{1}{2}mv^2 - 0 \text{ or } \int_0^{F/k} kx\,dx\cos 180^\circ + \int_0^{F/k} F\,dx = \frac{1}{2}mv^2$$

$$-\frac{F^2}{2k} + \frac{F^2}{k} = \frac{1}{2}mv^2 \quad \text{or} \quad v = \sqrt{\frac{F^2}{mk}}$$

Let x_0 be the maximum elongation

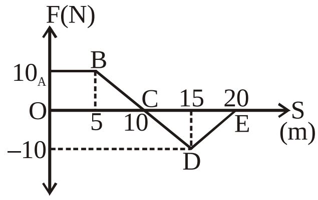
When elongation becomes maximum the block again comes to rest.

Therefore from work energy theorem we get,

$$W_{Spring} + W_{Force} = 0 - 0\,,\quad \int_0^{x_0} kxdx\cos 180^{o} + \int_0^{x_0} Fdx = 0$$

$$-\frac{1}{2}kx_0^2 + Fx_0 = 0\,,\qquad \therefore\ x_0 = \frac{2F}{k}$$

6. Force acting on a particle of mass 2kg varies with distance as shown in the figure. If speed of the particle at S = 0m be 5 m/s, calculate its speed at S = 20 m.



Soln. Total work done

= area OABCO – area CEDC

$= 10\times5+\frac{1}{2}\times10\times5 - \frac{1}{2}\times10\times10 = 75 - 50 = 25$ Joule

Work energy theorem

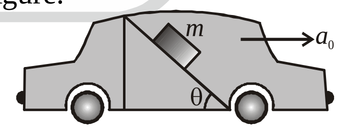
Total work done = change in K.E.

$$25 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad = \frac{1}{2}\times2\times v_2^2 - \frac{1}{2}\times2\times25$$

$$v_2^2 = 50 \quad \Rightarrow v_2 = 5\sqrt{2}\ \text{m/s}$$

7. A block of mass m stays at rest on an inclined plane inside a car which is moving with constant acceleration a_0. Calculate work done by friction on block as the car moves by a distance x.

Soln. Free body diagram of block in the frame of the car is shown in figure.

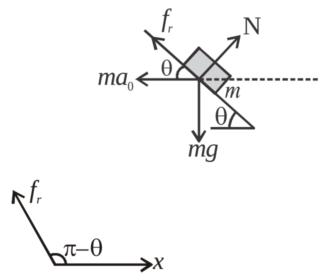


Block is in equilibrium.

$\therefore$ Along the plane

$$f_r + ma_0\cos\theta = mg\sin\theta$$

or $f_r = m(g\sin\theta - a_0\cos\theta)$



As the car moves through a distance 'x',
the block also moves the same distance.

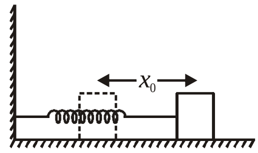
$\therefore$ Work done by friction

$$= \vec{f}_r\cdot\vec{x} \quad = f_r\,x\cos(\pi-\theta)$$

$$= -f_r\,x\cos\theta \quad = -mx(g\sin\theta - a_0\cos\theta)\cos\theta$$

8. A block of mass M is attached to one end of a spring of mass m and force constant K. The other end of spring is attached to a wall and the block lies on a smooth horizontal surface. The block is compressed against the spring through a distance x_0 and released. Calculate the speed of the block when it passes through the mean position.

Soln. Let the spring is uniformly stretched



$\therefore$ displacement of an element dx

which is at a distance x from the fixed end is

$$=\frac{x_0}{L}x; L \text{ is present length of spring.}$$

Therefore, if v be the speed of block at an instant then velocity of element dx.

$$=\frac{v}{L}x$$

$$\therefore \text{ K.E. of element} = \frac{1}{2}\cdot\left(\frac{m}{L}dx\right)\cdot\left(\frac{v}{L}x\right)^2$$

$$\therefore \text{ K.E. of spring} = \int_0^L \frac{1}{2}\left(\frac{m}{L}dx\right)\cdot\left(\frac{v}{L}x\right)^2 = \frac{1}{6}mv^2$$

Now, let v_0 be the speed of block when it passes through the origin.

$$\therefore \text{ From conservation of energy we get, } \frac{1}{2}kx_0^2 = \frac{1}{2}Mv_0^2 + \frac{1}{6}mv_0^2$$

$$\therefore \quad v_0 = x_0\sqrt{\frac{3K}{3M+m}}$$

9. A man is drawing water from a well with a bucket which leaks uniformly. The bucket when full weights 20 kg and when it arrives to the top only half the water remains. The depth of the surface of water is 20 meters. If g = 10 m/sec^2, what is the work done in lifting it up ?

Soln. When the bucket arrives at the top, the mass is 10 kg. Hence loss in mass = 20 – 10 = 10 kg. The depth of the well is 20 meters.

Therefore, mass lost per unit distance $=\frac{10}{20}=\frac{1}{2}$ kg.

Consider a point at a height x from the bottom of the well, (i.e., water surface)

At height x from the bottom, the bucket weight $=\left(20-\frac{x}{2}\right)$kg. The work done against the force during elementary displacement dx is $dW=\left(20-\frac{x}{2}\right)dx\cdot g$

Therefore, total workdone

$$W=\int_0^{20}\left(20-\frac{x}{2}\right)g\,dx = g\int_0^{20}\left[20\,dx-\frac{1}{2}x\,dx\right] = g\left[\left\{20\,x\right\}_0^{20}-\left\{\frac{x^2}{4}\right\}_0^{20}\right]$$

$$=g\left[400-100\right]=300\,g=300\times 10=\mathbf{3000\ J.}$$

10. A body of mass m is thrown at an angle α to the horizontal with an initial velocity v_0. Find the mean power delivered by gravity over the whole time of motion of the body and the instantaneous power of gravity as a function of time.

Soln. We know that $P_{\text{instan.}}=\vec{F}\cdot\vec{v}$

The velocity of the particle after time t is given by $\vec{v}=v_0\cos\alpha\,\hat{i}+(v_0\sin\alpha-gt)\,\hat{j}$ and $\vec{F}=-mg\,\hat{j}$.