CSIR-NET – MATHEMATICAL SCIENCES
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PART - B (Mathematical Sciences)

UNIT-1

21. Consider the function $\tan x$ on the set

$$S=\left\{x\in\mathbb{R}: x\geq 0,\ x\neq k\pi+\frac{\pi}{2} \text{ for any } k\in\mathbb{N}\cup\{0\}\right\}$$

We say that it has a fixed point in S if $\exists\, x\in S$ such that $\tan x = x$. Then,

(a) there is a unique fixed point.
(b) there is no fixed point.
(c) there are infinitely many fixed points.
(d) there are more than one but finitely many fixed points.

22. Define $f(x)=\dfrac{1}{\sqrt{x}}$ for $x>0$. Then f is uniformly continuous

(a) on $(0,\infty)$
(b) on $[r,\infty)$ for any $r>0$
(c) on $(0,r]$ for any $r>0$
(d) only on intervals of the form $[a,b]$ for $0<a<b<\infty$

23. Consider the subspaces W_1 and W_2 of $\mathbb{R}^3$ given by:

$W_1=\{(x,y,z)\in\mathbb{R}^3: x+y+z=0\}$, and

$W_2=\{(x,y,z)\in\mathbb{R}^3: x-y+z=0\}$

If W is a subspace of $\mathbb{R}^3$ such that

(i) $W\cap W_2=\text{span}\{(0,1,1)\}$

(ii) $W\cap W_1$ is orthogonal to $W\cap W_2$ with respect to the usual inner product of $\mathbb{R}^3$, then

(a) $W=\text{span}\{(0,1,-1),(0,1,1)\}$
(b) $W=\text{span}\{(1,0,-1),(0,1,-1)\}$
(c) $W=\text{span}\{(1,0,-1),(0,1,1)\}$
(d) $W=\text{span}\{(1,0,-1),(1,0,1)\}$

24. Let $C=\left[\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right]$ be a basis of $\mathbb{R}^2$ and $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x-2y\end{pmatrix}$. If $T[C]$ represents the matrix of T with respect to the basis C then which among the following is true ?

(a) $T[C]=\begin{bmatrix}-3 & -2\\ 3 & 1\end{bmatrix}$
(b) $T[C]=\begin{bmatrix}3 & -2\\ -3 & 1\end{bmatrix}$
(c) $T[C]=\begin{bmatrix}-3 & -1\\ 3 & 2\end{bmatrix}$
(d) $T[C]=\begin{bmatrix}3 & -1\\ -3 & 2\end{bmatrix}$

25. Let $W_1 = \{(u, v, w, x) \in \mathbb{R}^4 \mid u + v + w = 0,\ 2v + x = 0,\ 2u + 2w - x = 0\}$

and

$W_2 = \{(u, v, w, x) \in \mathbb{R}^4 \mid u + w + x = 0,\ u + w - 2x = 0,\ v - x = 0\}$

Then which among the following is true ?

(a) $\dim(W_1) = 1$ (b) $\dim(W_2) = 2$

(c) $\dim(W_1 \cap W_2) = 1$ (d) $\dim(W_1 + W_2) = 3$

26. Let A be an $n \times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + iI_n$. Then all eigenvalues of $(A - iI_n)B$ are :

(a) purely imaginary (b) of modulus one (c) real (d) of modulus less than one

27. Let $\{u_1, u_2, ..., u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ as column vectors. Let $M = (u_1, ..., u_k)$, $N = (u_{k+1}, ..., u_n)$ and p be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$. Then which of the following is true?

(a) Rank $(MPM^*) = k$ whenever $\alpha_i \neq \alpha_j\ 1 \leq i, j \leq k$

(b) Trace $(MPM^*) = \sum_{i=1}^{k} \alpha_i$

(c) Rank $(M^*N) = \min(k, n-k)$

(d) Rank $(MM^* + NN^*) < n$

28. Let $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function $B(a, b) = ab$. Which of the following is true ?

(a) B is a linear transformation. (b) B is a positive definite bilinear form.

(c) B is symmetric but not positive definite. (d) B is neither linear nor bilinear.

29. Consider the map $f : \mathbb{Q} \to \mathbb{R}$ defined by:

(i) $f(0) = 0$

(ii) $f(r) = \dfrac{p}{10^q}$, where $r = \dfrac{p}{q}$ with $p \in \mathbb{Z}, q \in \mathbb{Z}$ and gcd $(p, q) = 1$.

Then the map f is :

(a) one-to-one and onto. (b) not one-to-one, but onto.

(c) onto but not one-to-one. (d) neither one-to-one nor onto.

30. Let x be a real number such that $|x| < 1$. Which of the following is FALSE ?

(a) If $x \in \mathbb{Q}$, then $\sum_{m \geq 0} x^m \in \mathbb{Q}$ (b) If $\sum_{m \geq 0} x^m \in \mathbb{Q}$ then $x \in \mathbb{Q}$

(c) If $x \notin \mathbb{Q}$ then $\sum_{m \geq 0} m x^{m-1} \notin \mathbb{Q}$ (d) If $\sum_{m \geq 0} \dfrac{x^m}{m}$ converges in $\mathbb{R}$

31. Suppose that $\{x_n\}$ is a sequence of real numbers satisfying the following for every $\varepsilon > 0$, there exists n_0 such that $|x_{n+1} - x_n| < \varepsilon \ \forall\ n \ge n_0$. The sequence $\{x_n\}$ is :

(a) bounded but not necessarily Cauchy. (b) Cauchy but not necessarily bounded.
(c) convergent. (d) not necessarily bounded.

32. Let $A(n) = \int_n^{n+1} \frac{1}{x^3} dx$ for $n \ge 1$. For $c \in \mathbb{R}$ let $\lim_{n\to\infty} n^c A(n) = L$. Then :

(a) $L = 0$ if $c > 3$ (b) $L = 1$ if $c = 3$ (c) $L = 2$ if $c = 3$ (d) $L = \infty$ if $0 < c < 3$

UNIT-2

33. Consider the polynomials $p(z), q(z)$ in the complex variable z and let

$$I_{p,q} = \oint_\gamma p(z)\overline{q(z)}\,dz$$

where γ denotes the closed contour $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Then —

(a) $I_{z^m, z^n} = 0$ for all positive integers m, n with $m \ne n$.

(b) $I_{z^n, z^n} = 2\pi i$ for all positive integers n.

(c) $I_{p,1} = 0$ for all polynomials p

(d) $I_{p,q} = p(0)\overline{q(0)}$ for all polynomials p, q.

34. Let $\gamma(t) = 3e^{it}$, $0 \le t \le 2\pi$ be the positively oriented circle of radius 3 centered at the origin. The value of λ for which $\oint_\gamma \frac{\lambda}{z-2} dz = \oint_\gamma \frac{1}{z^2 - 5z + 4} dz$ is :

(a) $\lambda = -\frac{1}{3}$ (b) $\lambda = 0$ (c) $\lambda = \frac{1}{3}$ (d) $\lambda = 1$

35. The number of group homomorphisms from the alternating group A_5 to the symmetric group S_4 is :

(a) 1 (b) 12 (c) 20 (d) 6

36. Let $p \ge 23$ be a prime number such that the decimal expansion (base 10) of $\frac{1}{p}$ is periodic with period $p-1$ $\left(\text{that is, } \frac{1}{p} = 0\cdot\overline{a_1 a_2 \ldots a_{p-1}}\right)$ with $a_i \in \{0, 1, \ldots, 9\}$ for all i and for any $m, 1 \le m < p-1, \frac{1}{p} \ne 0\cdot\overline{a_1 a_2 \ldots a_m}$.

Let $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$ denote the multiplicative group of integers modulo p. Then which of the following is correct ?

(a) The order of $10 \in \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$ is a proper divisor of $(p-1)$.

(b) The order of $10 \in \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$ is $\frac{(p-1)}{2}$.

(c) The element $10 \in \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$ is a generator of the group $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$.

(d) The group $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$ is cyclic but not generated by the element 10.

37. Given integers *a* and *b*, let $N_{a,b}$ denote the number of positive integers $k < 100$ such that $k \equiv a \pmod 9$ and $k \equiv b \pmod{11}$. Then which of the following statements is correct ?

(a) $N_{a,b} = 1$ for all integers *a* and *b*.

(b) There exist integers *a* and *b* satisfying $N_{a,b} > 1$.

(c) There exist integers *a* and *b* satisfying $N_{a,b} = 0$.

(d) There exist integers *a* and *b* satisfying $N_{a,b} = 0$ and there exist integers *c* and *d* satisfying $N_{c,d} > 1$.

38. Let *X* be a topological space and *U* be a proper dense open subset of *X*. Pick the correct statement from the following :

(a) If *X* is connected then *U* is connected.

(b) If *X* is compact then *U* is compact.

(c) If $X \setminus U$ is compact then *X* is compact.

(d) If *X* is compact, then $X \setminus U$ is compact.

39. Let *R* denote the radius of convergence of the power series

$$\sum_{k=1}^{\infty} kx^k$$

Then,

(a) $R > 0$ and the series is convergent on $[-R, R]$.

(b) $R > 0$ and the series converges at $x = -R$ but does not converge at $x = R$.

(c) $R > 0$ and the series does not converge outside $(-R, R)$.

(d) $R = 0$

40. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function and let Image $(f) = \{w \in \mathbb{C} : \exists\, z \in \mathbb{C}$ such that $f(z) = w\}$. Then,

(a) The interior of Image (f) is empty.

(b) Image (f) intersects every line passing through the origin.

(c) There exists a disc in the complex plane, which is disjoint from Image (f).

(d) Image (f) contains all its limit points.

UNIT-3

41. Let $u(x, t)$ be a function that satisfies the PDE $u_{xx} - u_{tt} = e^x + 6t$, $x \in \mathbb{R}$, $t > 0$ and the initial conditions

$u(x, 0) = \sin(x),\ u_t(x, 0) = 0$ for every $x \in \mathbb{R}$. Here subscripts denote partial derivatives corresponding to the variables indicated. Then the value of $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is :

(a) $e^{\pi/2}\left(1+\frac{1}{2}e^{\pi/2}\right)+\left(\frac{\pi^3+4}{8}\right)$ (b) $e^{\pi/2}\left(1+\frac{1}{2}e^{\pi/2}\right)+\left(\frac{\pi^3-4}{8}\right)$

(c) $e^{\pi/2}\left(1-\frac{1}{2}e^{\pi/2}\right)-\left(\frac{\pi^3+4}{8}\right)$ (d) $e^{\pi/2}\left(1-\frac{1}{2}e^{\pi/2}\right)-\left(\frac{\pi^3-4}{8}\right)$

42. Let $u(x,t)$ satisfy the IVP : $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x \in \mathbb{R},\ t > 0$

$$u(x, 0) = \begin{cases} 1 \ ; & 0 \le x \le 1 \\ 0 \ ; & \text{elsewhere} \end{cases}$$

Then the value of $\lim_{t \to 0^+} u(1, t)$ equals:

(a) e (b) π (c) 1/2 (d) 1

43. Let $f(x)$ be a polynomial of unknown degree taking the values

x	0	1	2	3
$f(x)$	2	7	13	16

All the fourth divided differences are –1/6. Then the coefficient of x^3 is :

(a) $\frac{1}{3}$ (b) $-\frac{2}{3}$ (c) 16 (d) –1

44. Consider the functional $J[y] = \int_0^2 (1 - y'^2)^2\, dx$ defined on $\{y \in C[0, 2] : y$ is piecewise C^1 and $y(0) = y(2) = 0\}$. Let y_e be a minimizer of the above functional. Then y_e has :

(a) a unique corner point. (b) two corner points.

(c) more than two corner points. (d) no corner points.

45. If ϕ is the solution of $\int_0^x (1 - x^2 + t^2)\phi(t)\, dt = \frac{x^2}{2}$, then $\phi\left(\sqrt{2}\right)$ is equal to:

(a) $\sqrt{2}e^{\sqrt{2}}$ (b) $\sqrt{2}e^{2}$ (c) $\sqrt{2}e^{2\sqrt{2}}$ (d) $2e^4$

46. Consider the two dimensional motion of a mass *m* attached to one end of a spring whose other end is fixed. Let *k* be the spring constant. The kinetic energy *T* and the potential energy *V* of the system are given by :

$T = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right)$ and $V = \frac{1}{2}kr^2$, $\left[\text{where } \dot{r} = \frac{dr}{dt} \text{ and } \dot{\theta} = \frac{d\theta}{dt} \text{ with } t \text{ as time}\right]$.

Then which of the following statements is correct ?

(a) r is an ignorable coordinate.

(b) θ is not an ignorable coordinate.

(c) $r^2\dot{\theta}$ remains constant throughout the motion.

(d) $r\dot{\theta}$ remains constant throughout the motion.

47. If $y_1(x)$ and $y_2(x)$ are two solutions of the differential equation:

$$(\cos x)\, y'' + (\sin x)\, y' - (1 + e^{-x^2})\, y = 0 \;\forall\; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

with $y_1(0) = \sqrt{2},\; y_1'(0) = 1,\; y_2(0) = -\sqrt{2},\; y_2'(0) = 2$

then the Wronskian of $y_1(x)$ and $y_2(x)$ at $x = \frac{\pi}{4}$ is:

(a) $3\sqrt{2}$ (b) 6 (c) 3 (d) $-3\sqrt{2}$

48. The critical point (0, 0) for the system

$$x'(t) = x - 2y + y^2 \sin(x)$$

$$y'(t) = 2x - 2y - 3y\cos(y^2)$$

is a :

(a) stable spiral point (b) unstable spiral point

(c) saddle point (d) stable node

UNIT-4

49. To test the hypotheses H_0 against H_1 using the test statistic T, the proposed test procedure is not to support H_0 if T is large. Based on a given sample, the p-value of the test statistic is computed to be 0.05 assuming that the distribution of T is $N(0,1)$ under H_0. If the distribution of T under H_0 is the t-distribution with 10 degrees of freedom instead, the p-value will be:

(a) 0.05 (b) $< 0.05 - \frac{1}{100}$ (c) $0.05 - \frac{1}{100}$ (d) > 0.05

50. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n independent observations from a bivariate continuous distribution. Let r_p be the product moment correlation coefficient and r_s be the rank correlation coefficient computed based on these n observations. Which of the following statements is correct ?

(a) $r_p \geq 0$ implies $r_s \geq 0$ (b) $r_s \geq 0$ implies $r_p \geq 0$

(c) $r_p = 1$ implies $r_s = 1$ (d) $r_s = 1$ implies $r_p = 1$

51. Consider a linear model:

$$Y_i = \begin{cases} \theta_1 + \theta_2 + \varepsilon_i & \text{for} \quad i = 1, 2 \\ \theta_1 - \theta_3 + \varepsilon_i & \text{for} \quad i = 3, 4 \end{cases}$$

where ε_i's are independent with $\mathbb{E}(\varepsilon_i) = 0$, $\text{Var}(\varepsilon_i) = \sigma^2 > 0$ for $i = 1, \ldots, 4$ and $\theta_1, \ldots, \theta_3 \in \mathbb{R}$. Which of the following parametric functions is estimable ?

(a) $\theta_1 + \theta_3$ (b) $\theta_2 - \theta_3$ (c) $\theta_2 + \theta_3$ (d) $\theta_1 + \theta_2 + \theta_3$

52. If $X \sim N_p(0, 1)$ and $A_{p \times p}$ is an independent matrix with rank $(A) = k < p$, then which of the following statement is correct ?

(a) $\dfrac{X'AX}{X'X} \sim \dfrac{k}{p} F_{k,p}$ (b) $\dfrac{X'AX}{X'X} \sim \dfrac{k}{p-k} F_{k,p-k}$

(c) $\dfrac{X'AX}{X'X} \sim \beta\left(\dfrac{k}{2}, \dfrac{p}{2}\right)$ (d) $\dfrac{X'AX}{X'X} \sim \beta\left(\dfrac{k}{2}, \dfrac{p-k}{2}\right)$

53. A sample of size $n (\geq 2)$ is drawn from a population of $N (\geq 3)$ units using *PPSWR* sampling scheme, where p_i is the probability of selecting *i*-th unit in a draw, $0 < p_i < 1 \ \forall \ i = 1, \ldots, N$, and $\sum_{i=1}^{N} p_i = 1$. Then the inclusion probability π_{ij} is :

(a) $1 - p_i^n - p_j^n + (p_i + p_j)^n$ (b) $1 - (p_i + p_j - p_i p_j)^n$

(c) $1 - (1 - p_i)^n - (1 - p_j)^n - (p_i + p_j)^n$ (d) $1 - (1 - p_i)^n - (1 - p_j)^n + (1 - p_i - p_j)^n$

54. In a 2^4 experiment with two blocks and factors A, B, C and D, one block contains the following treatment combinations *a, b, c, ad, bd, cd, abc, abcd*. Which of the following effects is confounded ?

(a) ABC (b) ABD (c) BCD (d) ABCD

55. In an airport, domestic passengers and international passengers arrive independently according to Poisson processes with rates 100 and 70 per hour, respectively. If it is given that the total number of passengers (domestic and international) arriving in that airport between 9:00 AM and 11:00 AM on a particular day was 520, then what is the conditional distribution of the number of domestic passengers arriving in this period ?

(a) Poisson (200) (b) Poisson (100)

(c) Binomial $\left(520, \dfrac{10}{17}\right)$ (d) Binomial $\left(520, \dfrac{7}{17}\right)$

56. Let $X \geq 0$ be a random variable on (Ω, F, P) with $\mathbb{E}(X) = 1$. Let $A \in F$ be an event with $0 < P(A) < 1$. Which of the following defines another probability measure on (Ω, F) ?

(a) $Q(B) = P(A \cap B) \ \forall \ B \in F$ (b) $Q(B) = P(A \cup B) \ \forall \ B \in F$

(c) $Q(B) = \mathbb{E}(XI_B) \ \forall \ B \in F$ (d) $Q(B) = \begin{cases} P(A \mid B) & \text{if} \quad P(B) > 0 \\ 0 & \text{if} \quad P(B) = 0 \end{cases}$

57. Let *X* and *Y* be i.i.d. random variables uniformly distributed on (0, 4). Then $P(X > Y \mid X < 2Y)$ is:

(a) $\frac{1}{3}$ (b) $\frac{5}{6}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

58. Suppose $\{X_n\}$ is a Markov Chain with 3 states and transition probability matrix

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then which of the following statements is true ?

(a) $\{X_n\}$ is irreducible

(b) $\{X_n\}$ is recurrent

(c) $\{X_n\}$ does not admit a stationary probability distribution

(d) $\{X_n\}$ has an absorbing state

59. Suppose $X \sim$ Cauchy (0, 1). Then the distribution of $\frac{1-X}{1+X}$ is :

(a) Uniform (0, 1) (b) Normal (0, 1)

(c) Double exponential (0, 1) (d) Cauchy (0, 1)

60. Given the observations 0.8, 0.71, 0.9, 1.2, 1.68, 1.4, 0.88, 1.62 from the uniform distribution on $(\theta - 0.2, \theta + 0.8)$ with $-\infty < \theta < \infty$, which of the following is a maximum likelihood estimate for θ ?

(a) 0.7 (b) 0.9 (c) 1.1 (d) 1.3

PART - C (Mathematical Sciences)

UNIT-1

61. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function given by, $f(x, y) = (x^3 + 3xy^2 - 15x - 12y, x + y)$.

Let $S = \{(x, y) \in \mathbb{R}^2 : f$ is locally invertible at $(x, y)\}$. Then,

(a) $S = \mathbb{R}^2 \backslash \{(0, 0)\}$ (b) S is open in $\mathbb{R}^2$ (c) S is dense in $\mathbb{R}^2$ (d) $\mathbb{R}^2 \backslash S$ is countable

62. Let $X = N$, the set of positive integers. Consider the metrics d_1, d_2 on X given by:

$$d_1(m, n) = |m - n|, m, n \in X$$

$$d_2(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|, m, n \in X$$

Let X_1, X_2 denote the metric spaces $(X, d_1), (X, d_2)$ respectively. Then,

(a) X_1 is complete (b) X_2 is complete

(c) X_1 is totally bounded (d) X_2 is totally bounded

63. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that satisfies $T^2 = T - I_n$. Then which of the following are true ?

(a) T is invertible (b) $T - I_n$ is not invertible

(c) T has a real eigen value (d) $T^3 = -I_n$

64. Let $M = \begin{bmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 3 \end{bmatrix}$.

Then which of the following are true ?

(a) Both systems $MX = b_1$ and $MX = b_2$ are inconsistent.

(b) Both systems $MX = b_1$ and $MX = b_2$ are consistent.

(c) The system $MX = b_1 - b_2$ is consistent.

(d) The systems $MX = b_1 - b_2$ is inconsistent.

65. Let $M = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{bmatrix}$. Given that 1 is an eigenvalue of M, then which among the following are correct ?

(a) The minimal polynomial of M is $(X-1)(X+4)$.

(b) The minimal polynomial of M is $(X-1)^2(X+4)$.

(c) M is not diagonalizable.

(d) $M^{-1} = \frac{1}{4}(M + 3I)$

66. Let A be a real matrix with characteristic polynomial $(X-1)^3$. Pick the correct statements from below:

(a) A is necessarily diagonalizable.

(b) If the minimal polynomial of A is $(X-1)^3$, then A is diagonalizable.

(c) Characteristic polynomial of A^2 is $(X-1)^3$.

(d) If A has exactly two Jordan blocks, then $(A-1)^2$ is diagonalizable.

67. Let P_3 be the vector space of polynomials with real coefficients and of degree at most 3. Consider the linear map $T : P_3 \to P_3$ defined by $T(p(x)) = p(x+1) + p(x-1))$. Which of the following properties does the matrix of T (with respect to the standard basis $B = \{1, x, x^2, x^3\}$ of P_3) satisfy ?

(a) $\det T = 0$

(b) $(T - 2I)^4 = 0$ but $(T-2I)^3 \neq 0$

(c) $(T-2I)^3 = 0$ but $(T-2I)^2 \neq 0$

(d) 2 is an eigenvalue with multiplicity 4.

68. Let M be an $n \times n$ Hermitian matrix of rank k, $k \neq n$. If $\lambda \neq 0$ is an eigenvalue of M with corresponding unit column vector u, with $Mu = \lambda u$, then which of the following are true ?

(a) $\text{rank}(M - \lambda uu^*) = k - 1$

(b) $\text{rank}(M - \lambda uu^*) = k$

(c) $\text{rank}(M - \lambda uu^*) = k + 1$

(d) $(M - \lambda uu^*)^n = M^n - \lambda^n uu^*$

69. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as follows. If $v = (x_1, x_2)$, $w = (y_1, y_2)$ belong to $\mathbb{R}^2$ define $B(u, w) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. Let $v_0 = (1, 0)$ and let $W = \{v \in \mathbb{R}^2 : B(v_0, v) = 0\}$. Then W —

(a) is not a subspace of $\mathbb{R}^2$
(b) equals $\{(0, 0)\}$
(c) is the y-axis
(d) is the line passing through (0, 0) and (1, 1)

70. Consider the Quadratic forms:

$Q_1(x, y) = xy$

$Q_2(x, y) = x^2 + 2xy + y^2$

$Q_3(x, y) = x^2 + 3xy + 2y^2$

on $\mathbb{R}^2$. Choose the correct statements from below :

(a) Q_1 and Q_2 are equivalent
(b) Q_1 and Q_3 are equivalent
(c) Q_2 and Q_3 are equivalent
(d) All are equivalent

71. Let $\{u_n\}_{n\geq 1}$ be a sequence of real numbers satisfying the following conditions:

(1) $(-1)^n u_n \geq 0$, for all $n \geq 1$

(2) $|u_{n+1}| < \dfrac{|u_n|}{2}$, for all $n \geq 13$

Which of the following statements are necessarily true ?

(a) $\sum_{n\geq 1} u_n$ does not converge in $\mathbb{R}$.

(b) $\sum_{n\geq 13} u_n$ converges to zero.

(c) $\sum_{n\geq 13} u_n$ converges to a non-zero real number.

(d) If $|u_{n-1}| < \dfrac{|u_n|}{2}$, for all $2 \leq n \leq 13$, then $\sum_{n\geq 1} u_n$ is a negative real number.

72. Let S be an infinite set. Which of the following statements are true ?

(a) If there is an injection from S to $\mathbb{N}$, then S is countable.

(b) If there is a surjection from S to $\mathbb{N}$, then S is countable.

(c) If there is an injection from $\mathbb{N}$ to S, then S is countable.

(d) If there is a surjection from $\mathbb{N}$ to S, then S is countable.

73. Let p_n denote the n-th prime number, when we enumerate the prime numbers in the increasing order. For example, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on. Let $S = \{s_n = p_{n+1} - p_n \mid n \in \mathbb{N}, n \geq 1\}$. Then which of the following are correct ?

(a) $\sup S = \infty$

(b) $\limsup_{n\to\infty} s_n = \infty$

(c) $\inf S < \infty$ and $\inf S = 1$

(d) $\limsup_{n\to\infty} s_n \geq 2$

74. For $n \geq 1$, consider the sequence of functions:

$$f_n(x) = \frac{1}{2nx+1},\ g_n(x) = \frac{x}{2nx+1}$$

on the open interval (0, 1). Consider the statements:

(1) The sequence $\{f_n\}$ converges uniformly on (0, 1).

(2) The sequence $\{g_n\}$ converges uniformly on (0, 1).

Then,

(a) (1) is true

(b) (1) is false

(c) (1) is false and (2) is true

(d) Both (1) and (2) are true.

75. Suppose that $\{f_n\}$ is a sequence of continuous real valued functions on [0, 1] satisfying the following:

(1) $\forall\ x \in \mathbb{R}, \left\{f_n(x)\right\}$ is a decreasing sequence.

(2) The sequence $\{f_n\}$ converges uniformly to 0.

Let $g_n(x) = \sum_{k=1}^{n} (-1)^k f_k(x)\ \forall\ x \in \mathbb{R}$. Then —

(a) $\{g_n\}$ is Cauchy with respect to the sup norm.

(b) $\{g_n\}$ is uniformly convergent.

(c) $\{g_n\}$ need not converge pointwise.

(d) $\exists M > 0$ such that $|g_n(x)| \leq M,\ \forall\ n \in \mathbb{N},\ \forall\ x \in \mathbb{R}$.

76. Given $f : \left[\frac{1}{2}, 2\right] \to \mathbb{R}$, a strictly increasing function, we put $g(x) = f(x) + f\left(\frac{1}{x}\right), x \in [1, 2]$. Consider a partition P of [1, 2] and let $U(P, g)$ and $L(P, g)$ denote the upper Riemann sum and lower Riemann sum of g. Then,

(a) for a suitable f we can have $U(P, g) = L(P, g)$.

(b) for a suitable f we can have $U(P, g) \neq L(P, g)$.

(c) $U(P, g) \geq L(P, g)$ for all choices of f.

(d) $U(P, g) < L(P, g)$ for all choices of f.

77. Let f be a real valued continuously differentiable function of (0, 1). Set $g = f' + if$, where $i^2 = -1$ and f' is the derivative of f. Let $a, b \in (0,1)$ be two consecutive zeros of f. Which of the following statements are necessarily true ?

(a) If $g(a) > 0$, then g crosses the real line from upper half plane to lower half plane at a.

(b) If $g(a) > 0$, then g crosses the real line from lower half plane to upper half plane at a.

(c) If $g(a)g(b) \neq 0$, then $g(a), g(b)$ have the same sign.

(d) If $g(a)g(b) \neq 0$, then $g(a), g(b)$ have opposite signs.

78. Let A be an invertible real $n \times n$ matrix. Define a function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $F(x, y) = \langle Ax, y \rangle$, where $\langle x, y \rangle$ denotes the inner product of x and y. Let $DF(x, y)$ denote the derivative of F at (x, y) which is a linear transformation from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Then,

(a) If $x \neq 0$, then $DF(x, 0) \neq 0$

(b) If $y \neq 0$, then $DF(0, y) \neq 0$

(c) If $(x, y) \neq (0, 0)$ then $DF(x, y) \neq 0$

(d) If $x = 0$ or $y = 0$, then $DF(x, y) = 0$

UNIT-2

79. For any group G, let $\text{Aut}(G)$ denote the group of automorphisms of G. Which of the following are true ?

(a) If G is finite, then $\text{Aut}(G)$ is finite.

(b) If G is cyclic, then $\text{Aut}(G)$ is cyclic.

(c) If G is infinite, then $\text{Aut}(G)$ is infinite.

(d) If $\text{Aut}(G)$ is isomorphic to $\text{Aut}(H)$, where G and H are two groups, then G is isomorphic to H.

80. Let G be a group with the following property:
Given any positive integers m, n are r there exist elements g and h in G such that order $(g) = m$, order $(h) = n$ and order $(gh) = r$. Then which of the following are necessarily true ?

(a) G has to be an infinite group.

(b) G cannot be a cyclic group.

(c) G has infinitely many cyclic subgroups.

(d) G has to be a non-abelian group.

81. Let R be the ring $\mathbb{C}[x]/(x^2+1)$. Pick the correct statements from below:

(a) $\dim_{\mathbb{C}} R = 3$

(b) R has exactly two prime ideals

(c) R is a UFD

(d) (x) is a maximal ideal of R

82. Let $f(x) = x^7 - 105x + 12$. Then which of the following are correct ?

(a) $f(x)$ is reducible over $\mathbb{Q}$.

(b) There exists an integer m such that $f(m) = 105$.

(c) There exists an integer m such that $f(m) = 2$.

(d) $f(m)$ is not a prime number for any integer m.

83. Let $\alpha = \sqrt[5]{2} \in \mathbb{R}$ and $\xi = \exp\left(\frac{2\pi i}{5}\right)$. Let $K = \mathbb{Q}(\alpha\xi)$. Pick the correct statements from below:

(a) There exists a field automorphism σ of $\mathbb{C}$ such that $\sigma(K) = K$ and $\sigma \neq id$.

(b) There exists a field automorphism σ of $\mathbb{C}$ such that $\sigma(K) \neq K$.

(c) There exists a finite extension E of $\mathbb{Q}$ such that $K \subseteq E$ and $\sigma(K) \subseteq E$ for every field automorphism

σ of E.

(d) For all field automorphisms σ of K, $\sigma(\alpha\xi) = \alpha\xi$.

84. Let $X = \{(x_i)_{i\geq 1} : x_i \in \{0,1\} \text{ for all } i \geq 1\}$ with the metric $d((x_i),(y_i)) = \sum_{i\geq 1} |x_i - y_i| 2^{-i}$. Let $f : X \to [0,1]$ be the function defined by $f(x_i)_{i\geq 1} = \sum_{i\geq 1} x_i\, 2^{-i}$. Choose the correct statements from below:

(a) f is continuous (b) f is onto (c) f is one-to-one (d) f is open

85. Let A be a subset of $\mathbb{R}$ satisfying $A = \bigcap_{n\geq 1} V_n$, where for each $n \geq 1, V_n$ is an open dense subset of $\mathbb{R}$. Which of the following are correct ?

(a) A is a non-empty set. (b) A is countable.

(c) A is uncountable. (d) A is dense in $\mathbb{R}$.

86. Let H denote the upper half plane, that is, $H = \{z = x + iy : y > 0\}$. For $z \in H$, which of the following are true ?

(a) $\dfrac{1}{z} \in H$ (b) $\dfrac{1}{z^2} \in H$ (c) $\dfrac{-z}{z+1} \in H$ (d) $\dfrac{z}{2z+1} \in H$

87. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Then which of the following statements are true ?

(a) If $|f(z)| \leq 1$ for all $z \in \mathbb{C}$, then f' has infinitely many zeros in $\mathbb{C}$.

(b) If f is onto, then the function $f(\cos z)$ is onto.

(c) If f is onto, then the function $f(e^z)$ is onto.

(d) If f is one-one, then the function $f(z^4 + z + 2)$ is one-one.

88. Consider the entire functions $f(z) = 1 + z + z^{20}$ and $g(z) = e^z$, $z \in \mathbb{C}$. Which of the following statements are true ?

(a) $\lim_{|z|\to\infty} |f(z)| = \infty$

(b) $\lim_{|z|\to\infty} |g(z)| = \infty$

(c) $f^{-1}([z \in \mathbb{C} : |z| \leq \mathbb{R}\})$ is bounded for every $R > 0$.

(d) $g^{-1}(\{z \in \mathbb{C} : |z| \leq \mathbb{R}\})$ is bounded for every $R > 0$.

89. Which of the following statements are true ?

(a) $\tan z$ is an entire function. (b) $\tan z$ is a meromorphic function on $\mathbb{C}$.

(c) $\tan z$ has an isolated singularity at ∞. (d) $\tan z$ has a non-isolated singularity at ∞.

90. Let $a_1 < a_2 < \cdots a_{51}$ be given distinct natural numbers such that $1 \le a_i \le 100$ for all $i = 1, 2, \ldots, 51$. Then which of the following are correct ?

(a) There exist *i* and *j* with $1 \le i < j \le 51$ satisfying a_i divides a_j

(b) There exist *i* with $1 \le i \le 51$ such that a_i is an odd integer.

(c) There exists *j* with $1 \le j \le 51$ such that a_j is an even integer.

(d) There exist $i < j$ such that $|a_i - a_j| > 51$.

UNIT-3

91. Let $u(x, t)$ be a function that satisfies the PDE : $u_t + uu_x = 1,\ x \in \mathbb{R},\ t > 0$ and the initial condition $u\left(\frac{t^2}{4}, t\right) = \frac{t}{2}$. Then the IVP has :

(a) only one solution.
(b) two solutions.
(c) an infinite number of solutions.
(d) solutions none of which is differentiable on the characteristic base curve.

92. Let $f : [0.1] \to [0, 1]$ be twice continuously differentiable function with a unique fixed point $f(x_*) = x_*$. For a given $x_0 \in (0, 1)$ consider the iteration $x_{n+1} = f(x_n)$ for $n \ge 0$. If $L = \max_{x \in [0,1]} |f'(x)|$, then which of the following are true ?

(a) If $L < 1$, then x_n converges to x_*

(b) x_n converges to x_* provided $L \ge 1$

(c) The error $e_n = x_n - x_*$ satisfies $|e_{n+1}| < L|e_n|$.

(d) If $f'(x_*) = 0$, then $|e_{n+1}| < C|e_n|^2$ for some $C > 0$.

93. Let $u(x)$ satisfy the boundary value problem

$$(BVP)\begin{cases} u'' + u' = 0, & x \in (0, 1) \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

Consider the finite difference approximation to (BVP)

$$(BVP)_n \begin{cases} \dfrac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + \dfrac{U_{j+1} - U_{j-1}}{2h} = 0, \quad j = 1, \ldots, N-1 \\ U_0 = 0 \\ U_N = 1 \end{cases}$$

Here, U_j is an approximation to $u(x_j)$, where $x_j = jh,\ j = 0, \ldots, N$ is a partition of [0, 1] with $h = 1/N$ for some positive integer *N*. Then which of the following are true ?

(a) There exists a solution to $(BVP)_h$ of the form $U_j = ar^j + b$ for some $a, b \in \mathbb{R}$ with $r \neq 1$ and r satisfying $(2+h)r^2 - 4r + (2-h) = 0$.

(b) $U_j = \dfrac{(r^j - 1)}{(r^N - 1)}$, where r satisfies $(2+h)r^2 - 4r + (2-h) = 0$ and $r \neq 1$.

(c) u is monotonic in x

(d) U_j is monotonic in j.

94. Consider the functional $J[y] = \int_0^1 \left[(y')^2 - (y')^4 \right] dx$ subject to $y(0) = 0,\ y(1) = 0$. A broken extremal is a continuous extremal whose derivative has jump discontinuities at a finite number of points. Then which of the following statements are true ?

(a) There are no broken extremals and $y = 0$ is an extremal.

(b) There is a unique broken extremal.

(c) There exists more than one and finitely many broken extremals.

(d) There exist infinitely many broken extremals.

95. The extremals of the functional $J[y] = \int_0^1 [720\,x^2 y - (y'')^2]\,dx$, subject to $y(x) = y'(0) = y(1) = 0,\ y'(1) = 6$, are:

(a) $x^6 + 2x^3 - 3x^2$ (b) $x^5 + 4x^4 - 5x^3$ (c) $x^5 + x^4 - 2x^3$ (d) $x^6 + 4x^3 - 6x^2$

96. If ϕ is the solution of $\phi(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x-t) - 4(x-t)^2]\phi(t)\,dt$, then $\phi(\log 2)$ is equal to:

(a) 2 (b) 4 (c) 6 (d) 8

97. A characteristic number and the corresponding eigenfunction of the homogeneous Fredholm integral equation with kernel

$$K(x,t) = \begin{cases} x(t-1) & ; \quad 0 \le x \le t \\ t(x-1) & ; \quad t \le x \le 1 \end{cases}$$

are :

(a) $\lambda = -\pi^2, \phi(x) = \sin \pi x$ (b) $\lambda = -2\pi^2, \phi(x) = \sin 2\pi x$

(c) $\lambda = -3\pi^2, \phi(x) = \sin 3\pi x$ (d) $\lambda = -4\pi^2, \phi(x) = \sin 2\pi x$

98. Consider a point mass of mass m which is attached to a mass-less rigid rod of length a. The other end of the rod is made to move vertically such that its downward displacement from the origin at time t is given by

$$z(t) = z_0 \cos(\omega t)$$

The mass is moving in a fixed plane and its position vector at time t is given by:

$$\vec{r}(t) = (a \sin\theta(t),\ z(t) + a\cos\theta(t))$$

Then the equation of motion of the point mass is:

(a) $a\dfrac{d^2\theta}{dt^2}+(g+z_0\omega^2\cos(\omega t))\sin\theta=0$ (b) $a\dfrac{d^2\theta}{dt^2}+(g-z_0\omega\cos(\omega t))\sin\theta=0$

(c) $a\dfrac{d^2\theta}{dt^2}+(g+z_0^2\omega^2\cos(\omega t))\cos\theta=0$ (d) $a\dfrac{d^2\theta}{dt^2}+(g-z_0\omega^2\cos(\omega t))\cos\theta=0$

99. Three solutions of a certain second order non-homogeneous linear differential equation are :

$$y_1(x)=1+xe^{x^2},\ y_2(x)=(1+x)e^{x^2}-1,\ y_3(x)=1+e^{x^2}$$

Which of the following is/are general solution(s) of the differential equation ?

(a) $(C_1+1)y_1+(C_2-C_1)y_2-C_2y_3$, where C_1 and C_2 are arbitrary constants.

(b) $C_1(y_1-y_2)+C_2(y_2-y_3)$, where C_1 and C_2 are arbitrary constants.

(c) $C_1(y_1-y_2)+C_2(y_2-y_3)+C_3(y_3-y_1)$, where C_1, C_2 and C_3 are arbitrary constants.

(d) $C_1(y_1-y_3)+C_2(y_3-y_2)+y_1$, where C_1 and C_2 are arbitrary constants.

100. The method of variation of parameters to solve the differential equation $y''+p(x)y'+q(x)y=r(x)$, where $x\in I$ and $p(x), q(x), r(x)$ are non-zero continuous functions on an interval I, seeks a particular solution of the form $y_p(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$, where y_1 and y_2 are linearly independent solutions of $y''+p(x)y'+q(x)y=0$ and $v_1(x)$ and $v_2(x)$ are functions to be determined. Which of the following statements are necessarily true ?

(a) The Wronskian of y_1 and y_2 is never zero in I.

(b) v_1, v_2 and $v_1y_1+v_2y_2$ are twice differentiable.

(c) v_1 and v_2 may not be twice differentiable, but $v_1y_1+v_2y_2$ is twice differentiable.

(d) The solution set of $y''+p(x)y'+q(x)y=r(x)$ consists of function of the form $ay_1+by_2+y_p$, where $a, b\in\mathbb{R}$ are arbitrary constants.

101. Consider the eigenvalue problem,

$y''+\lambda y=0$ for $x\in(-1,1)$

$y(-1)=y(1)$

$y'(-1)=y'(1)$

Which of the following statements are true ?

(a) All eigenvalues are strictly positive.

(b) All eigenvalues are non-negative.

(c) Distinct eigenfunctions are orthogonal in $L^2[-1,1]$.

(d) The sequence of eigenvalues is bounded above.

102. Consider the IVP :

$xu_x+tu_t=u+1,\ x\in\mathbb{R},\ t\ge 0$

$u(x,t)=x^2,\ t=x^2$

Then,

(a) the solution is singular at (0, 0).

(b) the given space curve $(x, t, u) = (\xi, \xi^2, \xi^2)$ is not a characteristic curve at (0, 0).

(c) there is no base-characteristic curve in the (x, t) plane passing through (0, 0).

(d) a necessary condition for the IVP to have a unique C^1 solution at (0, 0) does not hold.

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(Answer Key) — PART - B (Mathematical Sciences)

21. (c)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (c)
26. (b)	**27.** (b)	**28.** (b)	**29.** (d)	**30.** (c)
31. (d)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (a)
36. (c)	**37.** (a)	**38.** (d)	**39.** (c)	**40.** (b)
41. (c)	**42.** (c)	**43.** (a)	**44.** (*)	**45.** (b)
46. (c)	**47.** (c)	**48.** (c)	**49.** (d)	**50.** (c)
51. (c)	**52.** (d)	**53.** (d)	**54.** (a)	**55.** (c)
56. (c)	**57.** (a)	**58.** (d)	**59.** (d)	**60.** (b)

(Answer Key) — PART - C (Mathematical Sciences)

61. (b), (c)	**62.** (a), (d)	**63.** (a), (d)	**64.** (a), (c)	**65.** (b), (c)
66. (c), (d)	**67.** (d)	**68.** (a), (d)	**69.** (*)	**70.** (b)
71. (c), (d)	**72.** (a), (d)	**73.** (a), (b), (c), (d)	**74.** (b), (c)	**75.** (a, b, d)
76. (a), (b), (c)	**77.** (b), (d)	**78.** (a), (b), (c)	**79.** (a)	**80.** (*)
81. (b)	**82.** (d)	**83.** (a), (b), (c), (d)	**84.** (a), (b)	**85.** (a, c, d)
86. (d)	**87.** (a), (b)	**88.** (a), (c)	**89.** (b), (d)	**90.** (a, b, c)
91. (b), (d)	**92.** (a)	**93.** (a), (b), (c), (d)	**94.** (d)	**95.** (*)
96. (a)	**97.** (a), (d)	**98.** (a)	**99.** (a), (d)	
100. (a), (b), (c)	**101.** (b), (c)	**102.** (a), (b), (c), (d)		