CAREER ENDEAVOUR
Best Institute for IIT-JAM, NET & GATE

CSIR-NET – MATHEMATICAL SCIENCES
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PART - B (Mathematical Sciences)

UNIT-1

1. Let $S=\{1,2,...,100\}$ and let $A=\{1,2,...,10\}$ and $B=\{41,42,...,50\}$. What is the total number of subsets of S, which have non-empty intersection with both A and B?

(a) $\frac{2^{100}}{2^{20}}$ (b) $\frac{100!}{10!10!}$ (c) $2^{80}(2^{10}-1)^2$ (d) $2^{100}-2(2^{10})$

Correct option is (c)

2. $\lim_{n\to\infty}\frac{1}{n}\left(1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}\right)$

(a) is equal to 0 (b) is equal to 1 (c) is equal to 2 (d) does not exist

Correct option is (b)

3. Consider the sequence $\{a_n\}_{n>1}$, where $a_n=3+5\left(-\frac{1}{2}\right)^n+(-1)^n\left(\frac{1}{4}+(-1)^n\frac{2}{n}\right)$.

Then the interval $\left(\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n\right)$ is given by:

(a) $(-2,8)$ (b) $\left(\frac{11}{4},\frac{13}{4}\right)$ (c) $(3,5)$ (d) $\left(\frac{1}{4},\frac{7}{4}\right)$

Correct option is (b)

4. Which of the following sets are countable?

(a) The set of all polynomials with rational coefficients.
(b) The set of all polynomials with real coefficients having rational roots.
(c) The set of all 2×2 real matrices with rational eigenvalues.
(d) The set of all real matrices whose row echelon form has rational entries.

Correct option is (a)

5. Let $f,g:\mathbb{R}\to\mathbb{R}$ be given by and $f(x)=x^2$ and $g(x)=\sin x$. Which of the following functions is uniformly continuous on $\mathbb{R}$?

(a) $h(x)=g(f(x))$ (b) $h(x)=g(x)f(x)$ (c) $h(x)=f(g(x))$ (d) $h(x)=f(x)+g(x)$

Correct option is (c)

6. Let $S_1 = \frac{1}{3} - \frac{1}{2}\times\frac{1}{3^2} + \frac{1}{3}\times\frac{1}{3^3} - \frac{1}{4}\times\frac{1}{3^4} + \cdots$ and $S_2 = \frac{1}{4} + \frac{1}{2}\times\frac{1}{4^2} + \frac{1}{3}\times\frac{1}{4^3} + \frac{1}{4}\times\frac{1}{4^4} + \cdots$

Which of the following identities is true ?

(a) $3S_1 = 4S_2$ (b) $4S_1 = 3S_2$ (c) $S_1 + S_2 = 0$ (d) $S_1 = S_2$

Correct option is (d)

7. Consider the two statements given below:

I. There exists a matrix $N \in \mathbb{M}_4(\mathbb{R})$ such that $\{(1,1,1,-1), (1,-1,1,1)\}$ is a basis of $\text{Row}(N)$ and $(1,2,1,4) \in \text{Null}(N)$

II. There exists a matrix $M \in \mathbb{M}_4(\mathbb{R})$ such that $\{(1,1,1,0)^T, (1,0,1,1)^T\}$ is a basis of $\text{Col}(M)$ and $(1,1,1,1)^T, (1,0,1,0)^T \in \text{Null}(M)$

Which of the following statements is true ?

(a) Statement-I is false and statement-II is true.

(b) Statement-I is true and statement-II is false.

(c) Both statement-I and statement-II are false.

(d) Both statement-I and statement-II are true.

Correct option is (a)

8. Let A and B be $n \times n$ matrices. Suppose the sum of the elements in any row of A is 2 and the sum of the elements in any column of B is 2. Which of the following matrices is necessarily singular ?

(a) $I - \frac{1}{2}BA^T$ (b) $I - \frac{1}{2}AB$ (c) $I - \frac{1}{4}AB$ (d) $I - \frac{1}{4}BA^T$

Correct option is (d)

9. Let A be a 4×4 matrix such that $-1, 1, 1, -2$ are its eigenvalues. If $B = A^4 - 5A^2 + 5I$, then trace $(A+B)$ equals

(a) 0 (b) –12 (c) 3 (d) 9

Correct option is (c)

10. Let $M = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 3 \end{bmatrix}$. Given that 1 is an eigenvalue of M, which of the following statements is true ?

(a) –2 is an eigenvalue of M

(b) 3 is an eigenvalue of M

(c) The eigen space of each eigen value has dimension 1

(d) M is diagonalizable

Correct option is (c)

11. Let $n > 1$ be a fixed natural number. Which of the following is an inner product on the vector space of $n \times n$ real symmetric matrices ?

(a) $\langle A, B\rangle = (\text{trace}(A))\,(\text{trace}(B))$ (b) $\langle A, B\rangle = \text{trace}(AB)$

(c) $\langle A, B\rangle = \text{determinant}(AB)$ (d) $\langle A, B\rangle = \text{trace}(A) + \text{trace}(B)$

Correct option is (b)

12. Let $V = \{A \in M_{3\times3}(\mathbb{R}) : A^t + A \in \mathbb{R}\cdot I\}$, where I is the identity matrix. Consider the quadratic form defined as $q(A) = \text{Trace}(A)^2 - \text{Trace}(A^2)$. What is the signature of this quadratic form ?

(a) $(+ + + +)$ (b) $(+ 0 0 0)$ (c) $(+ - - -)$ (d) $(- - - 0)$

Correct option is (a)

UNIT-2

13. Let $f(z)$ be a non-constant entire function and $z = x + iy$. Let $u(x, y), v(x, y)$ denote its real and imaginary parts respectively. Which of the following statements is false ?

(a) $u_x = v_y$ and $u_y = -v_x$ (b) $u_y = v_x$ and $u_x = -v_y$

(c) $|f'(x+iy)|^2 = u_x(x, y)^2 + v_x(x, y)^2$ (d) $|f'(x+iy)|^2 = u_y(x, y)^2 + v_y(x, y)^2$

Correct option is (b)

14. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and $O(\mathbb{D})$ be the space of all holomorphic function on $\mathbb{D}$. Consider the sets:

$$A = \left\{ f \in O(\mathbb{D}) : f\left(\frac{1}{n}\right) = \begin{cases} e^{-n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases};\ \text{for } n \geq 2, \right\}$$

$$B = \left\{ f \in O(\mathbb{D}) : f\left(\frac{1}{n}\right) = \frac{(n-2)}{(n-1)},\ n \geq 2 \right\}$$

Which of the following statements is true ?

(a) Both A and B are non-empty. (b) A is empty and B has exactly one element.

(c) A has exactly one element and B is empty. (d) Both A, B are empty.

Correct option is (b)

15. Let f be a rational function of a complex variable z given by:

$$f(z) = \frac{z^3 + 2z - 4}{z}$$

The radius of convergence of the Taylor series of f at $z = 1$ is:

(a) 0 (b) 1 (c) 2 (d) ∞

Correct option is (b)

16. Let γ be the positively oriented circle $\left\{z \in \mathbb{C} : |z| = \frac{3}{2}\right\}$. Suppose that $\int_\gamma \frac{e^{i\pi z}}{(z-1)(z-2i)^2} dz = 2\pi i C$. Then $|C|$ equals

(a) 2 (b) 5 (c) $\frac{1}{2}$ (d) $\frac{1}{5}$

Correct option is (d)

17. Let $S=\{n:1\le n\le 999;\ 3|n \text{ or } 37|n\}$. How many integers are there in the set $S^c=\{n:1\le n\le 999;\ n\notin S\}$?

(a) 639 (b) 648 (c) 666 (d) 990

Correct option is (b)

18. How many generators does a cyclic group of order 36 have ?

(a) 6 (b) 12 (c) 18 (d) 24

Correct option is (b)

19. Which of the following statements is necessarily true for a commutative ring R with unity ?

(a) R may have no maximal ideals.

(b) R can have exactly two maximal ideals.

(c) R can have one or more maximal ideals but no prime ideals.

(d) R has at least two prime ideals.

Correct option is (b)

20. Let (X,d) be a metric space and let $f:X\to X$ be a function such that $d(f(x),f(y))\le d(x,y)$ for every $x,y\in X$. Which of the following statements is necessarily true ?

(a) f is continuous (b) f is injective

(c) f is surjective (d) f is injective if and only if f is surjective

Correct option is (a)

UNIT-3

21. If $y(x)$ is a solution of the equation $4xy''+2y'+y=0$ satisfying $y(0)=1$. Then $y''(0)$ is equal to

(a) $\frac{1}{24}$ (b) $\frac{1}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$

Correct option is (b)

22. Consider the following two initial value ODEs

(A) $\frac{dx}{dt}=x^3,\ x(0)=1;$

(B) $\frac{dx}{dt}=x\sin x^2,\ x(0)=2$

Related to these ODEs, we make the following asserations.

(I) The solutions to (A) blows up in finite time.

(II) The solution to (B) blows up in finite time.

Which of the following statements is true ?

(a) Both (I) and (II) are true. (b) (I) is true but (II) is false.

(c) Both (I) and (II) are false. (d) (I) is false but (II) is true.

Correct option is (b)

23. Let $u(x, y)$ solve the Cauchy problem:

$$\frac{\partial u}{\partial y} - x\frac{\partial u}{\partial x} + u - 1 = 0, \text{ where } -\infty < x < \infty,\ y \geq 0 \text{ and } u(x, 0) = \sin x$$

Then $u(0,1)$ is equal to:

(a) $1-\frac{1}{e}$ (b) $1+\frac{1}{e}$ (c) $1-\frac{1-\sin e}{e}$ (d) $1+\frac{1-\sin e}{e}$

Correct option is (a)

24. Which of the following partial differential equations is NOT PARABOLIC for all $x, y \in \mathbb{R}$?

(a) $x^2\frac{\partial^2 u}{\partial x\,\partial y} - 2xy\frac{\partial u}{\partial y} + y^2 = 0$ (b) $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

(c) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ (d) $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Correct option is (a)

25. Let the solution to the initial value problem $y' = y - t^2 + 1,\ 0 \leq t \leq 2,\ y(0) = 0.5$ be computed using the Euler's method with step-length $h = 0.4$. If $y(0.8)$ and $w(0.8)$ denote the exact and approximate solutions at $t = 0.8$ then an error bound for Euler's method is given by:

(a) $0.2\,(0.5e^2 - 2)\,(e^{0.4} - 1)$ (b) $0.1(e^{0.4} - 1)$

(c) $0.2\,(0.5e^2 - 2)\,(e^{0.8} - 1)$ (d) $0.1(e^{0.8} - 1)$

Correct option is (c)

26. Which of the following is an extremal of the functional

$$J(y) = \int_{-1}^{1} (y'^2 - 2xy)\,dx$$

that satisfies the boundary conditions $y(-1) = -1$ and $y(1) = 1$?

(a) $-\frac{x^3}{5}+\frac{6x}{5}$ (b) $-\frac{x^5}{8}+\frac{9x}{8}$ (c) $-\frac{x^3}{6}+\frac{7x}{6}$ (d) $-\frac{x^3}{7}+\frac{8x}{7}$

Correct option is (c)

27. Let $a, b, c \in \mathbb{R}$ be such that the quadrature rule

$$\int_{-1}^{1} f(x)\,dx = a f(-1) + b f'(0) + c f'(1)$$

is exact for all polynomials of degree less than or equal to 2. Then $a+b+c$ equal to :

(a) 4 (b) 3 (c) 2 (d) 1

Correct option is (a)

28. A body moves freely in a uniform gravitational field. Which of the following statements is true?

(a) Stable equilibrium of the body is possible.

(b) Stable equilibrium of the body is not possible.

(c) Stable equilibrium of the body depends on the strength of the field.

(d) Equilibrium is metastable.

Correct option is (b)

29. Suppose that Y has exponential distribution with mean θ and that the conditional distribution of X given $Y = y$ is Normal with mean 0 and variance y, for all $y > 0$. Identify the characteristic function of X (defined as $\phi(t) = \mathbb{E}[e^{itX}]$ from the following.

(a) $e^{-\frac{\theta}{2}t^2}$ (b) $e^{-\frac{1}{2\theta}t^2}$ (c) $\dfrac{1}{1+\frac{1}{2}\theta t^2}$ (d) $\dfrac{1}{1+\frac{1}{2}t^2}$

Correct option is (c)

UNIT-4

30. **STATS QUESTIONS**

Correct option is ()

PART - C (Mathematical Sciences)

UNIT-1

1. Let (a_n) and (b_n) be two sequence of real numbers and E and F be two subsets of $\mathbb{R}$.

Let $E + F = \{a + b : a \in E, b \in F\}$. Assume that the right hand side is well defined in each of the following statements. Which of the following statements are true ?

(a) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(b) $\limsup\,(E+F) \le \limsup E + \limsup F$

(c) $\liminf_{n\to\infty}(a_n + b_n) \le \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

(d) $\liminf\,(E+F) = \liminf\, E + \limsup F$

Correct option is (a)

2. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ is a differentiable function. Consider the function $g(x) = e^x f(x)$. Which of the following are true ?

(a) If $\lim_{x\to\infty} f(x) = 0$ then $\lim_{x\to\infty} f'(x) = 0$

(b) If $\lim_{x\to\infty}(f(x) + f'(x)) = 0$ then $\lim\limits_{\substack{x\to\infty \\ y\to\infty}} \dfrac{g(x)-g(y)}{e^x - e^y} = 0$

(c) If $\lim_{x\to\infty} f'(x) = 0$ then $\lim_{x\to\infty} f(x) = 0$

(d) If $\lim_{x\to\infty}(f(x) + f'(x)) = 0$ then $\lim_{x\to\infty} f(x) = 0$

Correct options are (b), (d)

3. Let $A \subseteq \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Which of the following statements are true ?

(a) If A is closed then $f(A)$ is closed.

(b) If A is bounded then $f^{-1}(A)$ is bounded.

(c) If A is closed and bounded then $f(A)$ is closed and bounded.

(d) If A is bounded then $f(A)$ is bounded.

Correct options are (c), (d)

4. In which of the following cases does there exist a continuous and onto function $f : X \to Y$?

(a) $X = (0,1),\ Y = (0,1]$

(b) $X = [0,1],\ Y = (0,1]$

(c) $X = (0,1),\ Y = \mathbb{R}$

(d) $X = (0,2),\ Y = \{0,1\}$

Correct options are (a), (c)

5. Let Y be a non-empty bounded open subset of $\mathbb{R}^n$ and let $\bar{Y}$ denotes its closure. Let $\{U_j\}_{j\geq 1}$ be a collection of open sets in $\mathbb{R}^n$ such that $\bar{Y} \subseteq \bigcup_{j\geq 1} U_j$. Which of the following statements are true ?

(a) There exist finitely many positive integers $j_1, \dots, j_N$ such that $Y \subseteq \bigcup_{k=1}^{N} U_{j_k}$

(b) There exist a positive integer N such that $Y \subseteq \bigcup_{j=1}^{N} U_j$

(c) For every subsequence $j_1, j_2, \dots$ we have $Y \subseteq \bigcup_{k=1}^{\infty} U_{j_k}$

(d) There exists a subsequence $j_1, j_2, \dots$ such that $Y \subseteq \bigcup_{k=1}^{\infty} U_{j_k}$

Correct options are (a), (b)

6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^t f(x)\,dx = \int_t^1 f(x)\,dx$, for every $t \in [0,1]$. Then which of the following are necessarily true ?

(a) f is differentiable on (0, 1)

(b) f is monotonic on [0, 1]

(c) $\int_0^1 f(x)\,dx = 1$

(d) $f(x) > 0$ for all rationals $x \in [0,1]$

Correct options are (a), (b)

7. For non-negative integer $k \geq 1$ define $f_k(x) = \dfrac{x^k}{(1+x)^2}$ $\forall\ x \geq 0$. Which of the following statements are true?

(a) For each $k,\ f_k$ is a function of bounded variation on compact intervals.

(b) For every $k,\ \int_0^\infty f_k(x)\,dx < \infty$

(c) $\lim_{k\to\infty} \int_0^1 f_k(x)\,dx$ exists

(d) The sequence of functions f_k converge uniformly on [0, 1] as $k \to \infty$

Correct options are (a), (c)

8. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a bounded function such that for each $t \in \mathbb{R}$, the functions g_t and h_t given by $g_t(y) = f(t, y)$ and $h_t(x) = f(x, t)$ are non decreasing functions. Which of the following statements are necessarily true ?

(a) $k(x) = f(x, x)$ is a non-decreasing function.

(b) Number of discontinuous of f is at most countably infinite.

(c) $\lim_{(x, y)\to(+\infty,+\infty)} f(x, y)$ exists

(d) $\lim_{(x, y)\to(+\infty,-\infty)} f(x, y)$ exists

Correct options are (a), (c)

9. Let $f:\mathbb{R}^3 \to \mathbb{R}^2$ be a C^1 function with $f(0, 0, 0) = (0, 0)$. Let A denote the derivative of f at (0, 0, 0). Let $g:\mathbb{R}^3 \to \mathbb{R}$ be the function given by:

$$g(x, y, z) = xy + yz + zx + x + y + z$$

Let $h:\mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $h = (f, g)$.

In which of the following cases, will the function h admit a differentiable inverse in some open neighbourhood of (0, 0, 0) ?

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & 2 & 2 \\ 6 & 5 & 2 \end{pmatrix}$ (c) $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ (d) $A = \begin{pmatrix} 4 & 2 & 4 \\ 0 & 3 & 2 \end{pmatrix}$

Correct options are (c), (d)

10. Let $T: X \to Y$ be a bounded linear operator from a Banach space X to another Banach space Y. Which of the following conditions imply that T has a bounded inverse ?

(a) $\inf_{\|x\|=1} \|Tx\| = 0$

(b) $\inf_{\|x\|=1} \|Tx\| = 0$ and $T(X)$ is dense in Y

(c) $\inf_{\|x\|=1} \|Tx\| > 0$

(d) $\inf_{\|x\|=1} \|Tx\| > 0$ and $T(X)$ is dense in Y

Correct option is (d)

11. Let A be an $m \times n$ matrix such that the first r rows of A are linearly independent and the first s column of A are linearly independent, where $r < m$ and $s < n$. Which of the following statements are true ?

(a) The rank of A is at least $\max\{r, s\}$

(b) The submatrix formed by the first r row and first s columns of A has rank $\min\{r, s\}$

(c) If $r < s$, then there exists a row among rows $r+1, ..., m$ which together with the first r rows form a linearly dependent set.

(d) If $s < r$, then there exists a column among columns $s+1, ..., n$ which together with the first s columns form a linearly independent set.

Correct options are (a), (c)

12. Let A be an $m \times m$ matrix with real entries and let X be an $m \times 1$ vector of unknowns. Now consider the two statements given below:

I. There exists non-zero vector $b_1 \in \mathbb{R}^m$ such that the linear system $AX = b_1$ has NO solution.

II. There exist non-zero vectors $b_2, b_3 \in \mathbb{R}^m$, with $b_2 \neq cb_3$ for any $c \in \mathbb{R}$, such that the linear systems $AX = b_2$ and $AX = b_3$ have solutions.

Which of the following statements are true ?

(a) II is true whenever A is singular.
(b) I is true whenever A is singular.
(c) Both I and II can be true simultaneously.
(d) If $m = 2$, then at least one of I and II is false.

Correct options are (b), (c)

13. Let $M \in \mathbb{M}_n(\mathbb{R})$ such that $M \neq 0$ but $M^2 = 0$. Which of the following statements are true ?

(a) If n is even then $\dim(\mathrm{Col}(M)) > \dim(\mathrm{Null}\, M)$
(b) If n is even then $\dim(\mathrm{Col}(M)) \leq \dim(\mathrm{Null}\, M)$
(c) If n is odd then $\dim(\mathrm{Col}(M)) < \dim(\mathrm{Null}\, M)$
(d) If n is odd then $\dim(\mathrm{Col}(M)) > \dim(\mathrm{Null}\, M)$

Correct options are (b), (c)

14. Let A be an $n \times n$ matrix. We say that A is diagonalizable if there exists a nonsingular matrix P such that PAP^{-1} is a diagonal matrix. Which of the following conditions imply that A is diagonalizable ?

(a) There exists integer k such that $A^k = I$
(b) There exists integer k such that A^k is nilpotent.
(c) A^2 is diagonalizable
(d) A has n linearly independent eigenvectors.

Correct options are (a), (d)

15. It is known that $X = X_0 \in M_2(\mathbb{Z})$ is a solution of $AX - XA = A$ for some

$$A \in \left\{ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

Which of following values are NOT possible for the determinant of X_0 ?

(a) $\det(X_0) = 0$
(b) $\det(X_0) = 2$
(c) $\det(X_0) = 6$
(d) $\det(X_0) = 10$

Correct option is (d)

16. Which of the following are inner product on $\mathbb{R}^2$?

(a) $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + x_2 y_2$

(b) $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + 2x_2 y_2$

(c) $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$

(d) $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1y_1 - \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1 + x_2y_2$

Correct options are (b), (d)

17. Let X be a topological space and E be a subset of X. Which of the following statements are correct ?

(a) E is connected implies $\bar{E}$ is connected.

(b) E is connected implies ∂E is connected.

(c) E is path connected implies $\bar{E}$ is path connected.

(d) E is compact implies $\bar{E}$ is compact.

Correct option is (a)

18. Consider the system

$$\begin{aligned} 2x + ky &= 2 - k \\ kx + 2y &= k \\ ky + kz &= k - 1 \end{aligned}$$

In three unknowns and one real parameter k. For which of the following values of k is the system of linear equations consistent ?

(a) 1 (b) 2 (c) –1 (d) –2

Correct options are (a), (c)

UNIT-2

19. For any complex valued function f let D_f denote the set on which the function f satisfies Cauchy Riemann equations. Identify the functions for which D_f is equal to $\mathbb{C}$.

(a) $f(z) = \dfrac{z}{1+|z|}$

(b) $f(z) = (\cos\alpha x - \sin\alpha y) + i(\sin\alpha x + \cos\alpha y)$, where $z = x + iy$

(c) $f(z) = \begin{cases} e^{-1/z^4} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$

(d) $f(z) = x^2 + iy^2$, where $z = x + iy$

Correct option is (c)

20. Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane and let $\mathbb{D}$ be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Let R denote the set of points z_0 in $\mathbb{T}$ for which there exists a holomorphic function f in an open neighbourhood U_{z_0} of z_0 such that $f(z) = \sum_{n=0}^{\infty} z^{4n}$ in $U_{z_0} \cap \mathbb{D}$. Then R contains

(a) All points of $\mathbb{T}$ (b) Infinitely many points of $\mathbb{T}$

(c) All points of $\mathbb{T}$ except a finite set (d) No points of $\mathbb{T}$

Correct options are (b), (c)

21. Consider the function, $f(z)=\dfrac{(\sin z)^m}{(1-\cos z)^n}$ for $0<|z|<1$, where m, n are positive integers. Then $z=0$ is:

(a) A removable singularity if $m \ge 2n$
(b) A pole if $m < 2n$
(c) A pole if $m \ge 2n$
(d) An essential singularity for some values of m, n

Correct options are (a), (b)

22. Let f be an entire function such that $\left|zf(z)-1+e^z\right| \le 1+|z|$ for all $z \in \mathbb{C}$. Then,

(a) $f'(0)=1$
(b) $f'(0)=-\dfrac{1}{2}$
(c) $f''(0)=-\dfrac{1}{3}$
(d) $f''(0)=-\dfrac{1}{4}$

Correct options are (b), (c)

23. A positive integer n co-prime to 17, is called a primitive root modulo 17 if n^k-1 is not divisible by 17 for all k with $1 \le k < 16$. Let a, b be distinct positive integers between 1 and 16. Which of the following statements are true ?

(a) 2 is a primitive root modulo 17.
(b) If a is a primitive root modulo 17, then a^2 is not necessarily a primitive root modulo 17.
(c) If a, b are primitive roots modulo 17, then ab is a primitive root modulo 17.
(d) Product of primitive roots modulo 17 between 1 and 16 is congruent to 1 modulo 17.

Correct options are (b), (d)

24. For a positive integer n, let $\Omega(n)$ denote the number of prime factors of n, counted with multiplicity. For instance, $\Omega(3)=1,\ \Omega(6)=\Omega(9)=2$. Let $p>3$ be a prime number and let $N=p(p+2)(p+4)$. Which of the following statements are true ?

(a) $\Omega(N) \ge 3$
(b) There exist primes $p>3$ such that $\Omega(N)=3$
(c) p can never be the smallest prime divisor of N
(d) p can be the smallest prime divisor of N

Correct options are (a), (c)

25. Let p be a prime number and N_p be the number of pairs of positive integers (x, y) such that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{p^4}$$

Which among the following are possible values of N_p ?

(a) 0
(b) 4
(c) 5
(d) 9

Correct option is (d)

26. Let G be a group of order 24. Which of the following statements are necessarily true ?

(a) G has a normal subgroup of order 3.
(b) G is not a simple group.
(c) There exists an injective group homomorphism from G to S_8.
(d) G has a subgroup of index 4.

Correct options are (b), (d)

27. Which of the following statements are true ?

(a) All finite field extensions of $\mathbb{Q}$ are Galois.

(b) There exists a Galois extension of $\mathbb{Q}$ of degree 3.

(c) All finite field extensions of $\mathbb{F}_2$ are Galois.

(d) There exists a field extension of $\mathbb{Q}$ of degree 2 which is not Galois.

Correct options are (b), (c)

28. Let $f = a_0 + a_1X + \cdots a_nX^n$ be a polynomial with $a_i \in \mathbb{Z}$ for $0 \le i \le n$. Let p be a prime such that $p \mid a_i$ for all $1 < i \le n$ and p^2 does not divide a_n. Which of the following statements are TRUE ?

(a) f is always irreducible.

(b) f is always reducible.

(c) f can sometimes be irreducible and can sometimes be reducible.

(d) f can have degree 1.

Correct option is (c)

29. Which of the following statements are true about subsets of $\mathbb{R}^2$ with the usual topology?

(a) A is connected if and only if its closure $\overline{A}$ is connected.

(b) Intersection of two connected subsets is connected.

(c) Union of two compact subsets is compact.

(d) There are exactly two continuous functions from $\mathbb{Q}^2$ to the set $\{(0, 0), (1, 1)\}$.

Correct options are (c)

30. Consider $A = \{1, 1/2, 1/3, ..., 1/n, ... \mid n \in \mathbb{N}\}$ and $B = A \cup \{0\}$. Both the sets are endowed with subspace topology from $\mathbb{R}$. Which of the following statements are true ?

(a) A is a closed subset of $\mathbb{R}$

(b) B is a closed subset of $\mathbb{R}$

(c) A is homeomorphic to $\mathbb{Z}$, where $\mathbb{Z}$ has subspace topology from $\mathbb{R}$

(d) B is homeomorphic to $\mathbb{Z}$, where $\mathbb{Z}$ has subspace topology from $\mathbb{R}$

Correct options are (b), (c)

UNIT-3

31. Let u be a positive eigenfunction with eigenvalue λ for the boundary value problem $\ddot{u} + 2\dot{u} + a(t)u = \lambda u$, $\dot{u}(0) = 0 = \dot{u}(1)$, where $a : [0,1] \to (1, \infty)$ is a continuous function. Which of the following statements are possibly true ?

(a) $\lambda > 0$

(b) $\lambda < 0$

(c) $\int_0^1 (\dot{u})^2 dt = 2\int_0^1 u\dot{u}\, dt + \int_0^1 (a(t) - \lambda)u^2 dt$

(d) $\lambda = 0$

Correct options are (a), (c)

32. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-zero smooth vector field satisfying $\operatorname{div} f \neq 0$. Which of the following are necessarily true for the ODE $\dot{x} = f(x)$?

(a) There are no equilibrium points.
(b) There are no periodic solutions.
(c) All the solutions are bounded.
(d) All the solutions are unbounded.

Correct options are (a), (b)

33. Consider the 2nd order ODE $\ddot{x} + p(t)\dot{x} + q(t)x = 0$ and let x_1, x_2 be two solutions of this ODE in $[a, b]$. Which of the following statements are true for the Wronskian W of x_1, x_2 ?

(a) $W \equiv 0$ in (a, b) implies that x_1, x_2 are independent.

(b) W can change sign in (a, b).

(c) $W(t_0) = 0$ for some $t_0 \in (a, b)$ implies that $W \equiv 0$ in (a, b).

(d) $W(t_0) = 1$ for some $t_0 \in (a, b)$ implies that $W \equiv 1$ in (a, b).

Correct option is (c)

34. Which of the following expressions for $u = u(x, t)$ are solutions of $u_t - e^{-t} u_x + u = 0$ with $u(x, 0) = x$?

(a) $e^t(x + e^t - 1)$
(b) $e^{-t}(x - e^{-t} + 1)$
(c) $x - e^t + 1$
(d) xe^t

Correct option is (b)

35. Let $u(x, y)$ solve the partial differential equation (PDE)

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 3y^2 u = 0 \text{ with } u(x, 0) = e^{1/x}$$

Which of the following statements are true ?

(a) The PDE is not linear

(b) $u(1,1) = e^2$

(c) $u(1,1) = e^{-2}$

(d) The method of separation of variables can be utilized to compute the solution $u(x, y)$.

Correct options are (b), (d)

36. The values of a, b, c, d, e for which the function

$$f(x) = \begin{cases} a(x-1)^2 + b(x-2)^3 & -\infty < x \leq 2 \\ c(x-1)^2 + d & 2 \leq x \leq 3 \\ (x-1)^2 + e(x-3)^3 & 3 \leq x < \infty \end{cases}$$

is a cubic spline are:

(a) $a = c = 1, d = 0, b, e$ are arbitrary.
(b) $a = b = c = 1, d = 0, e$ is arbitrary.
(c) $a = b = c = d = 1, e$ is arbitrary.
(d) $a = b = c = d = e = 1$

Correct options are (a), (b)

37. Consider the Euler method for integration of the system of differential equations $\dot{x} = -y,\ \dot{y} = x$. Assume that $(x_i^n,\ y_i^n)$ are the points obtained for $i = 0, 1, \ldots, n^2$ using a time-step $h = \frac{1}{n}$ starting at the initial point $(x_0,\ y_0) = (1, 0)$. Which of the following statements are true ?

(a) The points $(x_i^n,\ y_i^n)$ lie on a circle of radius 1.

(b) $\lim_{n\to\infty}(x_n^n,\ y_n^n) = (\cos(1), \sin(1))$.

(c) $\lim_{n\to\infty}(x_2^n,\ y_2^n) = (1, 0)$.

(d) $(x_i^n)^2 + (y_i^n)^2 > 1$ for $i \geq 1$.

Correct options are (b), (c), (d)

38. Let $X = \{y \in C^1[0, \pi] : y(0) = 0 = y(\pi)\}$ and define $J : X \to \mathbb{R}$ by $J[y] = \int_0^\pi y^2(1 - y'^2)\,dx$. Which of the following statements are true ?

(a) $y \equiv 0$ is a local minimum for *J* with respect to the C^1 norm on *X*.

(b) $y \equiv 0$ is a local maximum for *J* with respect to the C^1 norm on *X*.

(c) $y \equiv 0$ is a local minimum for *J* with respect to the sup norm on *X*.

(d) $y \equiv 0$ is a local maximum for *J* with respect to the sup norm on *X*.

Correct option is (a)

39. Let *B* be the until ball in $\mathbb{R}^3$ centered at origin. The Euler-Lagrange equation corresponding to the functional $I(u) = \int_B (1 + |\nabla u|^2)^{1/2}\,dx$ is:

(a) $div\left(\dfrac{\nabla u}{(1 + |\nabla u|^2)^{1/2}}\right) = 0$

(b) $\dfrac{\nabla u}{(1 + |\nabla u|^2)^{1/2}} = 1$

(c) $|\nabla u| = 1$

(d) $\left(1 + |\nabla u|^2\right)\nabla u = \sum_{i,\,j=1}^{3} u_{x_i}\, u_{x_j}\, u_{x_i x_j}$

Correct options are (a), (d)

40. Let $K(x, y)$ be a kernel in [0, 1] × [0, 1], defined as $K(x, y) = \sin(2\pi x)\sin(2\pi y)$. Consider the integral operator $K(u)(x) = \int_0^1 u(y)\, K(x, y)\, dy$, where $u \in C([0, 1])$. Which of the following assertion on *K* are true?

(a) The null space of *K* is infinite dimensional.

(b) $\int_0^1 v(x)\, K(u)(x)\, dx = \int_0^1 K(v)(x)\, u(x)\, dx$ for all $u, v \in C([0, 1])$

(c) *K* has no negative eigenvalue.

(d) *K* has an eigenvalue greater than 3/4.

Correct options are (a), (b), (c)

41. Consider the integral equation:

$$\int_0^x (x-t)u(t)\,dt = x, \quad x \ge 0$$

for continuous functions u defined on $[0,\infty)$. The equation has:

(a) A unique bounded solution

(b) No solution

(c) A unique solution u such that $|u(x)| \le C(1+|x|)$ for some constant C.

(d) More than one solution u such that $|u(x)| \le C(1+|x|)$ for some constant C.

Correct option is (b)

42. A mass m with velocity v approaches a stationary mass M along the x-axis. The masses bounce of each other elastically. Assume that the motion takes place in one dimension along the x-axis, and v_f and V_f represents the final velocities of masses m and M along the x-axis. Which of the following are true ?

(a) $v_f = v, V_f = v$

(b) $v_f = 0, V_f = v$

(c) $v_f = \dfrac{(m-M)v}{m+M}, V_f = \dfrac{2mv}{m+M}$

(d) $v_f = \dfrac{mv}{m+M}, V_f = \dfrac{Mv}{m+M}$

Correct option is (c)

UNIT-4

Statistical Questions

43. Let X be a non-negative random variable with $E[X]=1$. Which of the following quantities is necessarily greater than or equal to 1 ?

(a) $E[X^4]$

(b) $(E[\cos X])^2 + (E[\sin X])^2$

(c) $E\left[\sqrt{X}\right]$

(d) $E[1/X]$

Correct options are (a), (d)