CSIR-NET – MATHEMATICAL SCIENCES
June-2019

PART - B (Mathematical Sciences)

UNIT-1

21. Which of the following sets is uncountable ?

(a) $\left\{x \in \mathbb{R} \mid \log(x) = \frac{p}{q} \text{ for some } p, q \in \mathbb{N}\right\}$ (b) $\left\{x \in \mathbb{R} \mid (\cos(x))^n + (\sin(x))^n = 1 \text{ for some } n \in \mathbb{N}\right\}$

(c) $\left\{x \in \mathbb{R} \mid x = \log\left(\frac{p}{q}\right) \text{ for some } p, q \in \mathbb{N}\right\}$ (d) $\left\{x \in \mathbb{R} \mid \cos(x) = \frac{p}{q} \text{ for some } p, q \in \mathbb{N}\right\}$

22. Consider a sequence: $\{a_n\}, a_n = (-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)$. Let $b_n = \sum_{k=1}^{n} a_k \ \forall\ n \in \mathbb{N}$. Then which of the following is true ?

(a) $\lim_{n\to\infty} b_n = 0$ (b) $\limsup_{n\to\infty} b_n > \frac{1}{2}$

(c) $\liminf_{n\to\infty} b_n < -\frac{1}{2}$ (d) $0 \le \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n \le \frac{1}{2}$

23. Which of the following is true ?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge (b) $\sum_{n=1}^{\infty} \frac{1}{n}$ converges

(c) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^2}$ converges (d) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^2}$ diverges

24. For $n \in \mathbb{N}$, which of the following is true ?

(a) $\sqrt{n+1} - \sqrt{n} > \frac{1}{\sqrt{n}}$ for all, except possibly finitely many n

(b) $\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}}$ for all, except possibly finitely many n

(c) $\sqrt{n+1} - \sqrt{n} > 1$ for all, except possibly finitely many n

(d) $\sqrt{n+1} - \sqrt{n} > 2$ for all, except possibly finitely many n

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and one-one function. Then which of the following is true ?

(a) f is onto (b) f is either strictly decreasing or strictly increasing

(c) there exists $x \in \mathbb{R}$ such that $f(x) = 1$ (d) f is unbounded

26. Let $g_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in [0, \infty)$. Which of the following is true as $n \to \infty$?

(a) $g_n \to 0$ pointwise but not uniformly (b) $g_n \to 0$ uniformly

(c) $g_n(x) \to x \ \forall \ x \in [0, \infty)$ (d) $g_n(x) \to \dfrac{x}{1+x^2} \ \forall \ x \in [0, \infty)$

27. Consider the vector space $\mathbb{P}_n$ of real polynomials in *x* of degree less than or equal to *n*. Define $T : \mathbb{P}_2 \to \mathbb{P}_3$ by $(Tf)(x) = \int_0^x f(t)\,dt + f'(x)$. Then the matrix representation of *T* with respect to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$ is:

(a) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1/2 & 0 \\ 0 & 2 & 0 & 1/3 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1/2 & 0 & 1/3 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1/2 \\ 0 & 2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

28. Let $P_A(x)$ denote the characteristic polynomial of a matrix *A*. Then for which of the following matrices, $P_A(x) - P_{A^{-1}}(x)$ is a constant ?

(a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

29. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

(a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

30. What is the rank of the following matrix ?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(a) 2 (b) 3 (c) 4 (d) 5

31. Let V denote the vector space of real valued continuous functions on the closed interval [0, 1]. Let W be the subspace of V spanned by $\{\sin(x), \cos(x), \tan(x)\}$. Then the dimensional of W over $\mathbb{R}$ is:

(a) 1 (b) 2 (c) 3 (d) Infinite

32. Let V be the vector space of polynomials in the variable t of degree at most 2 over $\mathbb{R}$. An inner product on V is defined by :

$$\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$$

for $f, g \in V$. Let $W = span\{1-t^2, 1+t^2\}$ and $W^{\perp}$ be the orthogonal complement of W in V. Which of the following conditions is satisfied for all $h \in W^{\perp}$?

(a) h is an even function, i.e., $h(t) = h(-t)$ (b) h is an odd function, i.e., $h(t) = -h(-t)$

(c) $h(t) = 0$ has a real solution (d) $h(0) = 0$

UNIT-2

33. Let C be the counter-clockwise oriented circle of radius 1/2 centered at $i = \sqrt{-1}$. Then the value of the contour integral $\oint_C \frac{dz}{z^4 - 1}$ is :

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $-\pi$ (d) π

34. Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$. Which of the following is false ?

(a) $f(\{z \in \mathbb{C} : |z| < 1\})$ is not an open set. (b) $f(\{z \in \mathbb{C} : |z| \le 1\})$ is not an open set.

(c) $f(\{z \in \mathbb{C} : |z| = 1\})$ is a closed set. (d) $f(\{z \in \mathbb{C} : |z| > 1\})$ is an unbounded open set.

35. Given a real number $a > 0$, consider the triangle Δ with vertices $0, a, a + ia$. If Δ is given the counter clockwise orientation, then the contour integral $\oint_{\Delta} \mathrm{Re}(z)\, dz$ (with $\mathrm{Re}(z)$ denoting the real part of z) is equal to,

(a) 0 (b) $i\frac{a^2}{2}$ (c) ia^2 (d) $i\frac{3a^2}{2}$

36. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\lim_{z \to 0} \left| f\left(\frac{1}{z}\right) \right| = \infty$. Then which of the following is true ?

(a) f is constant (b) f can have infinitely many zeros

(c) f can have at most finitely many zeros (d) f is necessarily nowhere vanishing

37. For any integer $n \geq 1$, let

$d(n) =$ Number of positive divisors of n

$v(n) =$ Number of distinct prime divisors of n

$\omega(n) =$ Number of prime divisors of n counted with multiplicity

[For example: If p is prime, then $d(p) = 2, v(p) = v(p^2) = 1, \omega(p^2) = 2$]

(a) If $n \geq 1000$ and $\omega(n) \geq 2$, then $d(n) > \log n$

(b) There exists n such that $d(n) > 3\sqrt{n}$

(c) For every n, $2^{v(n)} \leq d(n) \leq 2^{\omega(n)}$

(d) If $\omega(n) = \omega(m)$, then $d(n) = d(m)$

38. Consider the set of matrices:

$$G = \left\{ \begin{pmatrix} s & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z},\ s \in \{-1, +1\} \right\}$$

Then which of the following is true ?

(a) G forms a group under addition.

(b) G forms an abelian group under multiplication.

(c) Every element in G is diagonalisable over $\mathbb{C}$.

(d) G is a finitely generated group under multiplication.

39. Let R be a commutative ring with unity. Which of the following is true ?

(a) If R has finitely many prime ideals, then R is a field.

(b) If R has finitely many ideals, then R is finite.

(c) If R is a P.I.D., then every subring of R with unity is a P.I.D.

(d) If R is an integral domain which has finitely many ideals, then R is a field.

40. Let A be a non-empty subset of a topological space X. Which of the following statements is true ?

(a) If A is connected, then its closure $\overline{A}$ is not necessarily connected.

(b) If A is path connected, then its closure $\overline{A}$ is path connected.

(c) If A is connected, then its interior is not necessarily connected.

(d) If A is path connected, then its interior is connected.

UNIT-3

41. Let $y(x)$ be the solution of $x^2 y''(x) - 2y(x) = 0,\ y(1) = 1,\ y(2) = 1$. Then the value of $y(3)$ is:

(a) $\frac{11}{21}$ (b) 1 (c) $\frac{17}{21}$ (d) $\frac{11}{7}$

42. The positive values of λ for which the equation $y''(x) + \lambda^2 y(x) = 0$ has non-trivial solution satisfying $y(0) = y(\pi)$ and $y'(0) = y'(\pi)$ are:

(a) $\lambda = \frac{2n+1}{2},\ n = 1, 2, \ldots$ (b) $\lambda = 2n,\ n = 1, 2, \ldots$

(c) $\lambda = n,\ n = 1, 2, \ldots$ (d) $\lambda = 2n-1,\ n = 1, 2, \ldots$

43. Consider the PDE

$$P(x, y)\frac{\partial^2 u}{\partial x^2} + e^{x^2} e^{y^2} \frac{\partial^2 u}{\partial x \partial y} + Q(x, y)\frac{\partial^2 u}{\partial y^2} + e^{2x}\frac{\partial u}{\partial x} + e^y \frac{\partial u}{\partial y} = 0,$$

where P and Q are polynomials in two variables with real coefficients. Then which of the following is true for all choices of P and Q ?

(a) There exists $R > 0$ such that the PDE is elliptic in $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > R\right\}$.

(b) There exists $R > 0$ such that the PDE is hyperbolic in $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > R\right\}$.

(c) There exists $R > 0$ such that the PDE is parabolic in $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > R\right\}$.

(d) There exists $R > 0$ such that the PDE is hyperbolic in $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R\right\}$.

44. Let u be the unique solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, where $(x, t) \in (0,1) \times (0, \infty)$

$u(x, 0) = \sin \pi x, \ \ x \in (0,1)$ and $u(0, t) = u(1, t) = 0, \ \ t \in (0, \infty)$

Then which of the following is true ?

(a) There exists $(x, t) \in (0,1) \times (0, \infty)$ such that $u(x, t) = 0$.

(b) There exists $(x, t) \in (0,1) \times (0, \infty)$ such that $\frac{\partial u}{\partial t}(x, t) = 0$.

(c) The function $e^t u(x, t)$ is bounded for $(x, t) \in (0,1) \times (0, \infty)$.

(d) There exists $(x, t) \in (0,1) \times (0, \infty)$ such that $u(x, t) > 1$.

45. Consider solving the following system by Jacobi iteration scheme

$x + 2my - 2mz = 1$

$nx + y + nz = 2$

$2mx + 2my + z = 1$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n, satisfy

(a) $m + n = 3$ (b) $m > n$ (c) $m < n$ (d) $m = n$

46. Let $x^*(t)$ be the curve which minimizes the functional

$$J(x) = \int_0^1 \left[x^2(t) + \dot{x}^2(t) \right] dt$$

satisfying $x(0) = 0, \ x(1) = 1$. Then the value of $x^*\left(\frac{1}{2}\right)$ is :

(a) $\frac{\sqrt{e}}{1+e}$ (b) $\frac{2\sqrt{e}}{1+e}$ (c) $\frac{\sqrt{e}}{1+2e}$ (d) $\frac{2\sqrt{e}}{1+2e}$

47. If y is a solution of $y(x)-\int_0^x (x-t)y(t)\,dt = 1$, then which of the following is true ?

(a) y is bounded but not periodic in $\mathbb{R}$. (b) y is periodic in $\mathbb{R}$.

(c) $\int_{\mathbb{R}} y(x)\,dx < \infty$ (d) $\int_{\mathbb{R}} \frac{dx}{y(x)} < \infty$

48. Suppose a point mass m is attached to one end of a spring of spring constant k. The other end of the spring is fixed on a massless cart that is being moved uniformly on a horizontal plane by an external device with speed v_0. If the position q of the mass in the stationary system is taken as the generalized coordinate, then the Lagrangian of the system is :

(a) $\frac{m}{2}\dot{q}^2-\frac{k}{2}(q-v_0t)$ (b) $\frac{m}{2}\dot{q}^2-\frac{k}{2}(q-v_0t)^2$

(c) $\frac{m}{2}\dot{q}^2+\frac{k}{2}(q-v_0t)$ (d) $\frac{m}{2}\dot{q}^2+\frac{k}{2}(q-v_0t)^2$

UNIT-4

49. There are 30 questions in a certain multiple choice examination paper. Each question has 4 options and exactly one is to be marked by the candidate. Three candidates A, B, C mark each of the 30 questions at random independently. The probability that all the 30 answers of the three students match each other perfectly is:

(a) 60^{-4} (b) 30^{-4} (c) 4^{-30} (d) 4^{-60}

50. Let X_1, X_2, X_3, X_4, X_5 be i.i.d. random variables having a continuous distribution function. Then,

$P(X_1 > X_2 > X_3 > X_4 > X_5 \mid X_1 = \max(X_1, X_2, X_3, X_4, X_5))$ equals

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{4!}$ (d) $\frac{1}{5!}$

51. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then $\lim_{n\to\infty} p_{23}^{(n)}$ equals:

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 0 (d) 1

52. Consider the function $f(x)$ defined as $f(x) = ce^{-x^4}$, $x \in \mathbb{R}$. For what value of c is f a probability density function ?

(a) $\frac{2}{\Gamma(1/4)}$ (b) $\frac{4}{\Gamma(1/4)}$ (c) $\frac{3}{\Gamma(1/3)}$ (d) $\frac{1}{4\Gamma(4)}$

53. Suppose (X, Y) follows bivariate normal distribution with means μ_1, μ_2 standard deviations σ_1, σ_2 and correlation coefficient ρ, where all the parameters are unknown. Then testing $H_0 : \sigma_1 = \sigma_2$ is equivalent to testing the independent of —

(a) X and Y (b) X and $X - Y$ (c) $X + Y$ and Y (d) $X + Y$ and $X - Y$

54. A random sample of size 7 is drawn from a distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \dfrac{1+x^2}{3\theta(1+\theta^2)} & ; \quad -2\theta \le x \le \theta,\ \theta > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and the observations are: $12, -54, 26, -2, 24, 17, -39$. What is the maximum likelihood estimate of θ?

(a) 12 (b) 24 (c) 26 (d) 27

55. Let $X_1, X_2, \ldots$ be a sequence of independent normally distributed random variables with mean 1 and variance 1. Let N be a Poisson random variable with mean 2, independent of $X_1, X_2, \ldots$ Then, the variance of $X_1 + X_2 + \cdots + X_{N+1}$ is

(a) 3 (b) 4 (c) 5 (d) 9

56. There are two sets of observations on a random vector (X, Y). Consider a simple linear regression model with an intercept for regressing Y on X. Let $\hat{\beta}_i$ be the least squares estimate of the regression coefficient obtained from the i-th $(i = 1, 2)$ set consisting of n_i observations $(n_1, n_2 > 2)$. Let $\hat{\beta}_0$ be the least square of size $n_1 + n_2$. If it is known that $\hat{\beta}_1 > \hat{\beta}_2 > 0$, which of the following statements is true ?

(a) $\hat{\beta}_2 < \hat{\beta}_0 < \hat{\beta}_1$

(b) $\hat{\beta}_0$ may lie outside $(\hat{\beta}_2, \hat{\beta}_1)$, but it cannot exceed $\hat{\beta}_1 + \hat{\beta}_2$.

(c) $\hat{\beta}_0$ may lie outside $(\hat{\beta}_2, \hat{\beta}_1)$, but it cannot be negative.

(d) $\hat{\beta}_0$ can be negative.

57. Suppose $r_{1.23}$ and $r_{1.234}$ are sample multiple correlation coefficients of X_1 on X_2, X_3 and X_1 on X_2, X_3, X_4 respectively. Which of the following is possible ?

(a) $r_{1.23} = -0.3,\ r_{1.234} = 0.7$ (b) $r_{1.23} = 0.7,\ r_{1.234} = 0.3$

(c) $r_{1.23} = 0.3,\ r_{1.234} = 0.7$ (d) $r_{1.23} = 0.7,\ r_{1.234} = -0.3$

58. A sample of size $n = 2$ is drawn from a population of size $N = 4$ using probability proportional to size without replacement sampling scheme, where the probabilities proportional to size are:

i	:	1	2	3	4
p_i	:	0.4	0.2	0.2	0.2

The probability of inclusion of unit 1 in the sample is :

(a) 0.4 (b) 0.6 (c) 0.7 (d) 0.75

59. Suppose that $\begin{pmatrix} X \\ Y \end{pmatrix}$ has a bivariate density $f = \frac{1}{2} f_1 + \frac{1}{2} f_2$, where f_1 and f_2 are respectively, the densities of bivariate normal distributions $N(\mu_1, \Sigma)$ and $N(\mu_2, \Sigma)$, with $\mu_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mu_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\Sigma = I_2$, the 2×2 identity matrix. Then which of the following is correct ?

(a) X and Y are positively correlated.

(b) X and Y are negatively correlated.

(c) X and Y are uncorrelated but they are not independent.

(d) X and Y are independent.

60. The maximum value of the objective function $z = 5x + 2y$ under the linear constraints $x \ge 0,\ y \ge 0,\ x \ge y$ and $2 \le x + y \le 4$ is

(a) 14 (b) 20 (c) 25 (d) 27

PART - C (Mathematical Sciences)

UNIT-1

61. Let $\{a_n\}_{n \ge 0}$ be a sequence of positive real numbers. Then, for $K = \limsup_{n\to\infty} |a_n|^{1/n}$, which of the following are true ?

(a) If $K = \infty$, then $\sum_{n=0}^{\infty} a_n r^n$ is convergent for every $r > 0$.

(b) If $K = \infty$, then $\sum_{n=0}^{\infty} a_n r^n$ is not convergent for any $r > 0$.

(c) If $K = 0$, then $\sum_{n=0}^{\infty} a_n r^n$ is convergent for every $r > 0$.

(d) If $K = 0$, then $\sum_{n=0}^{\infty} a_n r^n$ is not convergent for any $r > 0$.

62. For $\alpha \in \mathbb{R}$, let $|\alpha|$ denote the greatest integer smaller than or equal to α. Define $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x, y) = \|x - y\|,\ x, y \in \mathbb{R}$. Then which of the following are true ?

(a) $d(x, y) = 0$ if and only if $x = y,\ x, y \in \mathbb{R}$.

(b) $d(x, y) = d(y, x),\ x, y \in \mathbb{R}$.

(c) $d(x, y) \le d(x, z) + d(z, y),\ x, y, z \in \mathbb{R}$.

(d) d is not a metric on $\mathbb{R}$.

63. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Then which of the following are true ?

(a) f is not one-one if the graph of f intersects some line parallel to X-axis in at least two points.

(b) f is one-one if the graph of f intersects any line parallel to the X-axis in at most one point.

(c) f is surjective if the graph of f intersects every line parallel to X-axis.

(d) f is not surjective if the graph of f does not intersect at least one line parallel to X-axis.

64. Let $f(x)=\int_1^{\infty}\frac{\cos t}{x^2+t^2}dt$. Then which of the following are true ?

(a) f is bounded on $\mathbb{R}$. (b) f is continuous on $\mathbb{R}$.

(c) f is not defined everywhere on $\mathbb{R}$. (d) f is not continuous on $\mathbb{R}$.

65. Suppose that $\{x_n\}$ is a sequence of positive reals. Let $y_n=\frac{x_n}{1+x_n}$. Then which of the following are true ?

(a) $\{x_n\}$ is convergent if $\{y_n\}$ is convergent. (b) $\{y_n\}$ is convergent if $\{x_n\}$ is convergent.

(c) $\{y_n\}$ is bounded if $\{x_n\}$ is bounded. (d) $\{x_n\}$ is bounded if $\{y_n\}$ is bounded.

66. Let $f(x)=\begin{cases} x\sin(1/x), & \text{for} \quad x\in(0,1] \\ 0, & \text{for} \quad x=0 \end{cases}$ and $g(x)=xf(x)$ for $0\le x\le 1$. Then which of the following are true ?

(a) f is of bounded variation. (b) f is not of bounded variation.

(c) g is of bounded variation. (d) g is not of bounded variation.

67. Let $a<c<b$, $f:(a,b)\to\mathbb{R}$ be continuous. Assume that f is differentiable at every point of $(a,b)\setminus\{c\}$ and f' has a limit at c. Then which of the following are true ?

(a) f is differentiable at c

(b) f need not be differentiable at c

(c) f is differentiable at c and $\lim_{x\to c} f'(x)=f'(c)$

(d) f is differentiable at c but $f'(c)$ is not necessarily $\lim_{x\to c} f'(x)$

68. Let $F:\mathbb{R}\to\mathbb{R}$ be a non-decreasing function. Which of the following can be the set of discontinuous of F ?

(a) $\mathbb{Z}$ (b) $\mathbb{N}$ (c) $\mathbb{Q}$ (d) $\mathbb{R}/\mathbb{Q}$

69. Let $f:\mathbb{R}^3\to\mathbb{R}^3$ be given by $f(x_1,x_2,x_3)=(e^{x_2}\cos x_1,\, e^{x_2}\sin x_1,\, 2x_1-\cos x_3)$.

Consider $E=\{(x_1,x_2,x_3)\in\mathbb{R}^3$: there exists an open subset U around (x_1,x_2,x_3) such that $f|_U$ is an open map$\}$. Then which of the following are true ?

(a) $E=\mathbb{R}^3$ (b) E is countable

(c) E is not countable but not $\mathbb{R}^3$ (d) $\left\{\left(x_1,x_2,\frac{\pi}{2}\right)\in\mathbb{R}^3 : x_1,x_2\in\mathbb{R}\right\}$ is a proper subset of E

70. Let X be a countable set. Then which of the following are true ?

(a) There exists a metric d on X such that (X,d) is complete.

(b) There exists a metric d on X such that (X,d) is not complete.

(c) There exists a metric d on X such that (X,d) is compact.

(d) There exists a metric d on X such that (X,d) is not compact.

71. Let $L(\mathbb{R}^n)$ be the space of $\mathbb{R}$-linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $\ker(T)$ denotes the kernel (null space) of T then which of the following are true ?

(a) There exists $T \in L(\mathbb{R}^5)\setminus\{0\}$ such that Range $(T) = \ker(T)$

(b) There does not exist $T \in L(\mathbb{R}^5)\setminus\{0\}$ such that Range $(T) = \ker(T)$

(c) There exists $T \in L(\mathbb{R}^6)\setminus\{0\}$ such that Range $(T) = \ker(T)$

(d) There does not exists $T \in L(\mathbb{R}^6)\setminus\{0\}$ such that Range $(T) = \ker(T)$

72. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T: V \to V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps $T_1 : V \to W$, $T_2 : W \to V$, where W is some finite dimensional vector space and such that

(a) Both T_1 and T_2 are onto.

(b) Both T_1 and T_2 are one to one.

(c) T_1 is onto, T_2 is one to one.

(d) T_1 is one to one, T_2 is onto.

73. Let $A = ((a_{ij}))$ be a 3×3 complex matrix. Identify the correct statements.

(a) $\det(((-1)^{i+j} a_{ij})) = \det A$

(b) $\det(((-1)^{i+j} a_{ij})) = -\det A$

(c) $\det(((\sqrt{-1})^{i+j} a_{ij})) = \det A$

(d) $\det(((\sqrt{-1})^{i+j} a_{ij})) = -\det A$

74. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t)\,dt,\ r(x) = \frac{d}{dx} p(x)$$

Let V denote the real vector space of all polynomials in x. Then which of the following are true ?

(a) q and r are linearly independent in V.

(b) q and r are linearly dependent in V.

(c) x^n belongs to the linear span of q and r.

(d) x^{n+1} belongs to the linear span of q and r.

75. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

(a) There exists matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity $n \times n$ matrix.

(b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalizable over $\mathbb{R}$ if and only if B is diagonalizable over $\mathbb{R}$.

(c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have same minimal polynomial.

(d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in $\mathbb{R}$.

76. Consider a matrix $A = (a_{ij})_{5\times 5}$, $1 \leq i, j \leq 5$ such that $a_{ij} = \dfrac{1}{n_i + n_j + 1}$, where $n_i, n_j \in \mathbb{N}$. Then in which of the following cases A is a positive definite matrix ?

(a) $n_i = i$ for all $i = 1, 2, 3, 4, 5$

(b) $n_1 < n_2 < \cdots < n_5$

(c) $n_1 = n_2 = \cdots = n_5$

(d) $n_1 > n_2 > \cdots > n_5$

77. Let $\langle\, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote the standard inner product on $\mathbb{R}^n$. For a non-zero $w \in \mathbb{R}^n$, define $T_w : \mathbb{R}^n \to \mathbb{R}^n$ by $T_w(v) = v - \dfrac{2\langle v, w\rangle}{\langle w, w\rangle} w$, for $v \in \mathbb{R}^n$. Which of the following are true ?

(a) $\det(T_w) = 1$

(b) $\langle T_w(v_1), T_w(v_2)\rangle = \langle v_1, v_2\rangle \; \forall\, v_1, v_2 \in \mathbb{R}^n$

(c) $T_w = T_w^{-1}$

(d) $T_{2w} = 2T_w$

78. Consider the matrix, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over the field $\mathbb{Q}$ of rationals. Which of the following matrices are of the form $P^t AP$ for a suitable 2×2 invertible matrix P over $\mathbb{Q}$? [Here, P^t denotes the transpose of P]

(a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

UNIT-2

79. Let $f(z) = (z^3 + 1)\sin z^2$ for $z \in \mathbb{C}$. Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and u, v are real valued functions. Then which of the following are true ?

(a) $u : \mathbb{R}^2 \to \mathbb{R}$ is infinitely differentiable.

(b) u is continuous but need not be differentiable.

(c) u is bounded.

(d) f can be represented by an absolutely convergent power series $\sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$.

80. Let $\operatorname{Re}(z), \operatorname{Im}(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$, respectively. Consider the domain $\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) > |\operatorname{Im}(z)|\}$ and let $f_n(z) = \log z^n$, where $n \in \{1, 2, 3, 4\}$ and where $\log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ defines the principal branch of logarithm. Then which of the following are true ?

(a) $f_1(\Omega) = \left\{ z \in \mathbb{C} : 0 \le |\operatorname{Im}(z)| < \dfrac{\pi}{4} \right\}$

(b) $f_2(\Omega) = \left\{ z \in \mathbb{C} : 0 \le |\operatorname{Im}(z)| < \dfrac{\pi}{2} \right\}$

(c) $f_3(\Omega) = \left\{ z \in \mathbb{C} : 0 \le |\operatorname{Im}(z)| < \dfrac{3\pi}{4} \right\}$

(d) $f_4(\Omega) = \left\{ z \in \mathbb{C} : 0 \le |\operatorname{Im}(z)| < \pi \right\}$

81. Consider the set, $F = \{ f : \mathbb{C} \to \mathbb{C} \mid f$ is an entire function, $|f'(z)| \le |f(z)|$ for all $z \in \mathbb{C} \}$

Then which of the following are true ?

(a) F is a finite set.

(b) F is an infinite set.

(c) $F = \{ \beta e^{\alpha z} : \beta \in \mathbb{C}, \alpha \in \mathbb{C} \}$

(d) $F = \{ \beta e^{\alpha z} : \beta \in \mathbb{C}, |\alpha| \le 1 \}$

82. Let $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ and $\omega \in D$. Define $F_\omega : D \to D$ by $F_\omega(z) = \dfrac{\omega - z}{1 - \bar{\omega} z}$. Then which of the following are true ?

(a) F is one to one

(b) F is not one to one

(c) F is onto

(d) F is not onto

83. Let $a \in \mathbb{Z}$ be such that $a = b^2 + c^2$, where $b, c \in \mathbb{Z} \setminus \{0\}$. Then '$a$' cannot be written as:

(a) pd^2, where $d \in \mathbb{Z}$ and p is a prime with $p \equiv 1 \pmod 4$.

(b) pd^2, where $d \in \mathbb{Z}$ and p is a prime with $p \equiv 3 \pmod 4$.

(c) pqd^2, where $d \in \mathbb{Z}$ and p, q are primes with $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$.

(d) pqd^2, where $d \in \mathbb{Z}$ and p, q are distinct primes with $p, q \equiv 3 \pmod 4$.

84. For any prime p, consider the group $G = GL_2(\mathbb{Z}/p\mathbb{Z})$. Then which of the following are true ?

(a) G has an element of order p.

(b) G has exactly one element of order p.

(c) G has no p-Sylow subgroups.

(d) Every element of order p is conjugate to a matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where $a \in (\mathbb{Z}/p\mathbb{Z})^*$.

85. Let $\mathbb{Z}[X]$ be the ring of polynomials over integers. Then the additive group $\mathbb{Z}[X]$ is :

(a) isomorphic to the multiplicative group $\mathbb{Q}^+$ of positive rational numbers.

(b) isomorphic to the group of rational numbers $\mathbb{Q}$ under addition.

(c) countable.

(d) uncountable.

86. Let $X = (0,1)$ be the open unit interval and $C(X, \mathbb{R})$ be the ring of continuous functions from X to $\mathbb{R}$. For any $x \in (0,1)$, let $I(x) = \{f \in C(X, \mathbb{R}) \mid f(x) = 0\}$. Then which of the following are true ?

(a) $I(x)$ is a prime ideal.

(b) $I(x)$ is a maximal ideal.

(c) Every maximal ideal of $C(X, \mathbb{R})$ is equal to $I(x)$ for some $x \in X$.

(d) $C(X, \mathbb{R})$ is an integral domain.

87. Let $n \in \mathbb{Z}$. Then which of the following are correct ?

(a) $X^3 + nX + 1$ is irreducible over $\mathbb{Z}$ for every n.

(b) $X^3 + nX + 1$ is reducible over $\mathbb{Z}$ if $n \in \{0, -2\}$.

(c) $X^3 + nX + 1$ is irreducible over $\mathbb{Z}$ if $n \notin \{0, -2\}$.

(d) $X^3 + nX + 1$ is reducible over $\mathbb{Z}$ for infinitely many n.

88. Let $\mathbb{F}_{27}$ denote the finite field of size 27. For each $\alpha \in \mathbb{F}_{27}$, we define

$A_\alpha = \{1, 1+\alpha, 1+\alpha+\alpha^2, 1+\alpha+\alpha^2+\alpha^3, \ldots.\}$

Then which of the following are true ?

(a) The number of $\alpha \in \mathbb{F}_{27}$ such that $|A_\alpha| = 26$ equals 12.

(b) $0 \in A_\alpha$ if and only if $\alpha \neq 0$.

(c) $|A_1| = 27$

(d) $\bigcap_{\alpha \in \mathbb{F}_{27}} A_\alpha$

89. Which of the following open covers of the open interval (0, 1) admit a finite subcover ?

(a) $\left\{\left(0, \frac{1}{2} - \frac{1}{n+1}\right) \cup \left(\frac{1}{n}, 1\right) : n \in \mathbb{N}\right\}$

(b) $\left\{\left(\frac{1}{n}, 1 - \frac{1}{n+1}\right) : n \in \mathbb{N}\right\}$

(c) $\left\{\left(\sin^2\left(\frac{n\pi}{100}\right), \cos^2\left(\frac{n\pi}{100}\right)\right) : n \in \mathbb{N}\right\}$

(d) $\left\{\left(\frac{1}{2}e^{-n}, 1 - \frac{1}{(n+1)^2}\right) : n \in \mathbb{N}\right\}$

90. Let X be a topogical space. Let A_1, A_2 be two dense subsets of X. If $E \subseteq X$, then which of the following are true ?

(a) $E \cap A_1$ is dense in E

(b) $E \cap A_1$ is dense in E if E is open

(c) $E \cap A_1 \cap A_2$ is dense in E if E is open

(d) $E \cap A_1 \cap A_2$ is dense in E if E, A_1, A_2 are all open.

UNIT-3

91. Let $y_1(x)$ be any non-trivial real valued solution of $y''(x) + xy(x) = 0,\ 0 < x < \infty$. Let $y_2(x)$ be the solution of $y''(x) + y(x) = x^2 + 2,\ y(0) = y'(0) = 0$. Then,

(a) $y_1(x)$ has infinitely many zeros.

(b) $y_2(x)$ has infinitely many zeros.

(c) $y_1(x)$ has finitely many zeros.

(d) $y_2(x)$ has finitely many zeros.

92. Consider the equation $y''(x) + a(x)y(x) = 0,\ a(x)$ is continuous function with period T. Let $\phi_1(x)$ and $\phi_2(x)$ be the basis for the solution satisfying $\phi_1(0) = 1,\ \phi_1'(0) = 0,\ \phi_2(0) = 0,\ \phi_2'(0) = 1$. Let $W(\phi_1, \phi_2)$ denote the Wronskian of ϕ_1 and ϕ_2. Then—

(a) $W(\phi_1, \phi_2) = 1$

(b) $W(\phi_1, \phi_2) = e^x$

(c) $\phi_1(T) + \phi_2'(T) = 2$ if the given differential equation has a non-trivial periodic solution with period T.

(d) $\phi_1(T) + \phi_2'(T) = 1$ if the given differential equation has a non-trivial periodic solution with period T.

93. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $f(x) = 0$ if and only if $x = \pm n^2$ where $n \in \mathbb{N}$. Consider the initial value problem: $y'(t) = f(y(t)),\ y(0) = y_0$. Then which of the following are true ?

(a) y is a monotone function for all $y_0 \in \mathbb{R}$.

(b) For any $y_0 \in \mathbb{R}$, there exists $M_{y_0} > 0$ such that $|y(t)| \le M_{y_0}$ for all $t \in \mathbb{R}$.

(c) There exists a $y_0 \in \mathbb{R}$, such that the corresponding solution y is unbounded.

(d) $\sup_{t,s \in \mathbb{R}} |y(t) - y(s)| = 2n + 1$ if $y_0 \in (n^2, (n+1)^2),\ n \ge 1$

94. The general solution $z = z(x, y)$ of $(x+y)\,z\,z_x + (x-y)\,z\,z_y = x^2 + y^2$ is :

(a) $F(x^2 + y^2 + z^2,\ z^2 - xy) = 0$ for arbitrary C^1 function F.

(b) $F(x^2 - y^2 - z^2,\ z^2 - 2xy) = 0$ for arbitrary C^1 function F.

(c) $F(x + y + z,\ z - 2xy) = 0$ for arbitrary C^1 function F.

(d) $F(x^3 - y^3 - z^3,\ z - 2x^2y^2) = 0$ for arbitrary C^1 function F.

95. Let u be the solution of the problem

$$\left.\begin{array}{lll} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & ; & (x, y) \in (0, \pi) \times (0, \pi), \\ u(0, y) = u(\pi, y) = 0 & ; & y \in (0, \pi), \\ u(x, 0) = 0,\ u(x, \pi) = \sin(2x) & ; & x \in (0, \pi). \end{array}\right\}$$

Then:

(a) $\max\{u(x, y) : 0 \le x, y \le \pi\} = 1$

(b) $u(x_0, y_0) = 1$ for some $(x_0, y_0) \in (0, \pi) \times (0, \pi)$

(c) $u(x, y) > -1$ for all $(x, y) \in (0, \pi) \times (0, \pi)$

(d) $\min\{u(x, y) : 0 \le x, y \le \pi\} > -1$

96. The values of a, b, c so that the truncation error in the formula:

$$\int_{-h}^{h} f(x)\,dx = ahf(-h) + bhf(0) + ahf(h) + ch^2 f'(-h) - ch^2 f'(h)$$

is minimum, are:

(a) $a = \dfrac{7}{15},\ b = \dfrac{16}{15},\ c = \dfrac{1}{15}$

(b) $a = \dfrac{7}{15},\ b = \dfrac{16}{15},\ c = \dfrac{-1}{15}$

(c) $a = \dfrac{7}{15},\ b = \dfrac{-16}{15},\ c = \dfrac{1}{15}$

(d) $a = \dfrac{7}{15},\ b = \dfrac{-16}{15},\ c = \dfrac{-1}{15}$

97. Consider the equation $x^2 + ax + b = 0$ which has two real roots α and β. Then which of the following iteration scheme converges when x_0 is chosen sufficiently close to α ?

(a) $x_{n+1} = -\dfrac{ax_n + b}{x_n}$, if $|\alpha| > |\beta|$

(b) $x_{n+1} = -\dfrac{x_n^2 + b}{a}$, if $|\alpha| > 1$

(c) $x_{n+1} = -\dfrac{b}{x_n + a}$, if $|\alpha| < |\beta|$

(d) $x_{n+1} = -\dfrac{x_n^2 + b}{a}$, if $2|\alpha| < |\alpha + \beta|$

98. Let u be the solution of

$$\left.\begin{array}{lll} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} & ; & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & ; & x \in \mathbb{R}, \\ u_t(x,0) = g(x) & ; & x \in \mathbb{R}, \end{array}\right\}$$

where f, g are in $C^2(\mathbb{R})$ and satisfy the following conditions.

(i) $f(x) = g(x) = 0$ for $x \le 0$ (ii) $0 < f(x) \le 1$ for $x > 0$

(iii) $g(x) > 0$ for $x > 0$ (iv) $\int_0^\infty g(x)\,dx < \infty$

Then, which of the following statements are true ?

(a) $u(x,t) = 0$ for all $x \le 0$ and $t > 0$ (b) u is bounded on $\mathbb{R} \times (0,\infty)$

(c) $u(x,t) = 0$ whenever $x + t < 0$ (d) $u(x,t) = 0$ for some (x,t) satisfying $x + t > 0$

99. Let $B = \left\{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \le 1\right\}$ and $C = \left\{(x,y) / x^2 + y^2 = 1\right\}$ and let f and g be continuous functions. Let u be the minimizer of the functional

$$J[v] = \iint_B \left(v_x^2 + v_y^2 - 2fv\right) dx\,dy + \int_C \left(v^2 - 2gv\right) ds$$

Then u is a solution of —

(a) $-\Delta u = f,\ \dfrac{\partial u}{\partial n} + u = g$ (b) $\Delta u = f,\ \dfrac{\partial u}{\partial n} - u = g$

(c) $-\Delta u = f,\ \dfrac{\partial u}{\partial n} = g$ (d) $\Delta u = f,\ \dfrac{\partial u}{\partial n} = g$

where $\dfrac{\partial u}{\partial n}$ denotes the directional derivative of u in the direction of the outward drawn normal at $(x, y) \in C$.

100. Consider the functional, $J[y] = \int_0^1 \left[(y'(x))^2 + (y'(x))^3\right] dx$, subject to $y(0) = 1$ and $y(1) = 2$. Then,

(a) There exists an extremal $y \in C^1([0,1]) \setminus C^2([0,1])$

(b) There exists an extremal $y \in C([0,1]) \setminus C^1([0,1])$

(c) Every extremal y belongs to $C^1([0,1])$

(d) Every extremal y belongs to $C^2([0,1])$

101. Consider the integral equation, $\phi(x) - \dfrac{e}{2}\int_{-1}^{1} xe^t\,\phi(t)\,dt = f(x)$, then —

(a) There exists a continuous function, $f : [-1,1] \to (0,\infty)$ for which solution exists.

(b) There exists a continuous function, $f : [-1,1] \to (-\infty, 0)$ for which solution exists.

(c) For $f(x) = e^{-x}(1 - 3x^2)$, a solution exists.

(d) For $f(x) = e^{-x}(x + x^3 + x^5)$, a solution exists.

102. Let (q, p) be canonical variables. Consider the following transformations:

(A) $(Q, P) = \left(\sqrt{2q}e^{\alpha}\cos p, \sqrt{2q}e^{-\alpha}\sin p\right), \alpha \in \mathbb{R}$

(B) $(Q, P) = (q\tan p, \log(\sin p))$

(C) $(Q, P) = \left(\frac{1}{p}, qp^2\right)$

Then,

(a) only the transformations given in (A) and (B) are canonical.

(b) only the transformations given in (B) and (C) are canonical.

(c) only the transformations given in (A) and (C) are canonical.

(d) all are canonical.

UNIT-4

103. STATS QUESTIONS ....

CSIR-NET – MATHEMATICAL SCIENCES
June-2019

(Answer Key) — PART - B (Mathematical Sciences)

21. (b)	**22.** (b)	**23.** (d)	**24.** (b)	**25.** (b)
26. (a)	**27.** (a or b)	**28.** (c)	**29.** (a)	**30.** (d)
31. (c)	**32.** (c)	**33.** (a)	**34.** (a)	**35.** (b)
36. (c)	**37.** (c)	**38.** (d)	**39.** (d)	**40.** (c)
41. (d)	**42.** (b)	**43.** (b)	**44.** (c)	**45.** (d)
46. (a)	**47.** (d)	**48.** (b)	**49.** (d)	**50.** (c)
51. (c)	**52.** (a)	**53.** (d)	**54.** (d)	**55.** (c)
56. (d)	**57.** (c)	**58.** (c)	**59.** (a)	**60.** (b)

(Answer Key) — PART - C (Mathematical Sciences)

61. (b), (c)	**62.** (b), (d)	**63.** (a), (b), (c), (d)	**64.** (a), (b)	**65.** (b), (c)
66. (b), (c)	**67.** (a), (c)	**68.** (a), (b), (c)	**69.** (c), (d)	
70. (a), (c) or (a), (b), (c), (d)				
71. (b), (c)	**72.** (c), (d)	**73.** (a), (c)	**74.** (a)	**75.** (d)
76. (a), (b), (d)	**77.** (b), (c)	**78.** (a), (b), (d)	**79.** (a), (d)	
80. (a), (b), (c), (d)				
81. (b), (d)	**82.** (a), (c)	**83.** (b), (c), (d)	**84.** (a), (d)	**85.** (a), (c)
86. (a), (b)	**87.** (b), (c)	**88.** (a), (b), (d)	**89.** (a), (c)	**90.** (b), (d)
91. (a), (d)	**92.** (a), (c)	**93.** (a), (b), (d)	**94.** (b)	**95.** (a), (c)
96. (a)	**97.** (a), (c), (d)	**98.** (b), (c)	**99.** (a)	**100.** (c, d)
101. (c), (d)	**102.** (d)			