CSIR-NET – MATHEMATICAL SCIENCES
NOV. 2020

PART - B (Mathematical Sciences)

1. Let $S = \{u_1, ..., u_k\}$ be a subset of non-zero vectors from $\mathbb{R}^n$. Now, consider the two statements given below:

I : If S is linearly dependent set in $\mathbb{R}^n$ then u_k is a linear combination of $u_1, ..., u_{k-1}$.

II : If S is linearly independent set in $\mathbb{R}^n$ then $k < n$.

Which of the following statements is true ?
(a) Statement-I is FALSE and statement-II is TRUE.
(b) Statement-I is TRUE and statement-II is FALSE.
(c) Both statement-I and statement-II are FALSE.
(d) Both statement-I and statement-II are TRUE.

2. Let a, b and c be distinct integers. Let A be the matrix.

$$A = \begin{pmatrix} a^2 & b^2 & c^2 \\ a^5 & b^5 & c^5 \\ a^{11} & b^{11} & c^{11} \end{pmatrix}$$

Which among the following is the set of all possible ranks of A ?
(a) $\{3\}$ (b) $\{2, 3\}$ (c) $\{1, 2, 3\}$ (d) $\{0, 1, 2, 3\}$

3. Which of the following is an inner product on the vector space of all real valued continuous functions on [0, 1]?

(a) $\langle f, g \rangle = \left| \int_0^1 f(t)\, g(t)\, dt \right|$ (b) $\langle f, g \rangle = \int_0^1 |f(t)\, g(t)|\, dt$

(c) $\langle f, g \rangle = f(0)\, g(0) + f(1)\, g(1)$ (d) $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$

4. $\lim\limits_{n\to\infty} \dfrac{((n+1)(n+2)\cdots(n+n))^{1/n}}{n}$

(a) is equal to $e/4$ (b) is equal to $4/e$ (c) is equal to e (d) does not exist

5. Suppose that A, B are two non-empty subsets of $\mathbb{R}$ and $C = A \cap B$. Which of the following conditions imply that C is empty ?
(a) A and B are open and C is compact. (b) A and B are open and C is closed.
(c) A and B are both dense in $\mathbb{R}$. (d) A is open and B is compact.

6. Let A be an $n \times n$ matrix of rank 1. Let $\alpha = \det(I + A)$, where I is the identity matrix and let $\beta = \text{trace } A$. Which of the following is true ?

(a) $\beta - \alpha = 1$ (b) $\alpha - \beta = 1$ (c) $\alpha < \beta + 1$ (d) $\alpha > \beta + 1$

7. An infinite binary word a is a string $(a_1 a_2 a_3 \ldots)$, where each $a_n \in [0, 1]$. Fix a word $s = (s_1 s_2 s_3 \ldots)$, where $s_n = 1$ if and only if n is prime. Let $S = \{a = (a_1 a_2 a_3 \ldots) \mid \exists\, m \in \mathbb{N} \text{ such that } a_n = s_n, \forall\, n \geq m\}$. What is the cardinality of S ?

(a) 1 (b) Finite but more than 1

(c) Countably infinite (d) Uncountable

8. Let $Y = \{1, 2, 3, \ldots, 100\}$, and let $h : Y \to Y$ be a strictly increasing function. The total number of functions $g : Y \to Y$ satisfying $g(h(j)) = h(g(j)), \forall\, j \in Y$ is:

(a) 0 (b) 100! (c) 100^{100} (d) 100^{98}

9. Let M be a 5×5 matrix with real entries such that Rank(M) = 3. Consider the linear system $Mx = b$. Let the row-reduced echelon form of the augmented matrix $[M\ b]$ be R and let $R[i, :]$ denote the i-th row of R. Suppose that the linear system admits a solution. Which of the following statements is necessarily true ?

(a) $R[3, :] = [0\ \ 1\ \ 0\ \ *\ \ *\ \ *]$ (b) $R[5, :] = [0\ \ 0\ \ 1\ \ 0\ \ *\ \ *]$

(c) $R[4, :] = [0\ \ 0\ \ 0\ \ 1\ \ *\ \ *]$ (d) $R[4, :] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]$

10. The sum of the infinite series:

$$S = \frac{1}{2} - \frac{1}{3 \times 1!} + \frac{1}{4 \times 2!} - \frac{1}{5 \times 3!} + \cdots$$

is equal to:

(a) $2 - \frac{1}{e}$ (b) $1 - \frac{2}{e}$ (c) $\frac{2}{e} - 1$ (d) $\frac{1}{e} - 2$

11. Let us define a matrix $A \in M_n(\mathbb{R})$ to be positive if for every column vector $v \in \mathbb{R}^n$ we have $\langle Av, v \rangle \geq 0$, where $\langle .,. \rangle$ is the standard inner product on $\mathbb{R}^n$. Let $A_{\alpha,\beta} = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \beta \end{pmatrix}$.

Let $S = \{(\alpha, \beta) \in \mathbb{R}^2 : A_{\alpha,\beta} \text{ is positive}\}$. Which of the following statements is true ?

(a) S is empty (b) $(\alpha, \beta) \in S$ if and only if $\alpha\beta > 0$

(c) $(\alpha, \beta) \in S$ if and only if $\alpha + \beta + 4 > 0$ (d) $S = \mathbb{R}^2$

12. Let f be a non-constant polynomial of degree k and let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Which of the following statements is necessarily true ?

(a) There always exists $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$.

(b) There is no $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$.

(c) There exists $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$ if k is even.

(d) There exists $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$ if k is odd.

13. Which of the following statements is NOT true ?

(a) The polynomial ring $\mathbb{Z}[x]$ is a Principal Ideal Domain (PID).

(b) The polynomial ring $\mathbb{Q}[x]$ is a Principal Ideal Domain (PID).

(c) The polynomial ring $\mathbb{Z}[x]$ is a Unique Factorization Domain (UFD).

(d) The polynomial ring $\mathbb{Q}[x]$ is a Unique Factorization Domain (UFD).

14. Let G be a group of order 2020. Which of the following statements is necessarily true ?

(a) G is not a simple group.
(b) G has exactly four proper subgroups.
(c) G is a cyclic group.
(d) G is abelian.

15. Let f be a holomorphic function on the disc $\{z \in \mathbb{C} : |z| < 2\}$. Assume that the only zero of f in the closed unit disc $\{z \in \mathbb{C} : |z| \le 1\}$ is a simple zero at the origin. Let γ be the positively oriented circle $\{z \in \mathbb{C} : |z| = 1\}$.

The integral $\int_\gamma \frac{dz}{f(z)}$ equals

(a) $2\pi i f'(0)$
(b) $2\pi i f''(0)$
(c) $\frac{2\pi i}{f'(0)}$
(d) $\frac{2\pi i}{f''(0)}$

16. Let f, g be entire functions such that $\lim_{z\to\infty} \frac{f(z)}{z^n} = \lim_{z\to\infty} \frac{g(z)}{z^n} = 1$ for some fixed positive integer n. Which of the following statements is true ?

(a) $f = g$

(b) $f - g$ is necessarily a polynomial of degree at most $n-1$

(c) There exist f, g with these properties such that $f - g$ is a polynomial of degree n.

(d) There exist f, g with these properties such that $f - g$ is not a polynomial.

17. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \left||z|^2 - 1\right|^2$. Which of the following statements is true ?

(a) f is complex differentiable at all complex numbers except for $z \in \{0, 1\}$.

(b) f is complex differentiable only at $z = 0$.

(c) f is complex differentiable only at $z = 1$.

(d) f is complex differentiable only for $z \in \{0, 1\}$.

18. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be analytic at ∞. If the function g defined by $g(w) = f\left(1/w\right)$ is analytic at 0 with an appropriate value given for $g(0)$. Which of the following statements is true ?

(a) Any non-constant polynomial is analytic at ∞.

(b) If f is analytic at ∞ then f is bounded.

(c) For any z_0 in $\mathbb{C}$, the function $f(z) = \frac{1}{e^{z-z_0}}$ is analytic at ∞.

(d) Any entire function can be extended to an analytic function at ∞.

19. The last two digits of 3^{2019} are:
(a) 27 (b) 37 (c) 57 (d) 67

20. Suppose $f:[0,1]\times[0,1]\to(0,1)\times(0,1)$ is a continuous non-constant function. Which of the following statements is NOT true ?
(a) Image of f is uncountable.
(b) Image of f is a path connected set.
(c) Image of f is a compact set.
(d) Image of f has non-empty interior.

21. Consider the ODE $t\dot{y}-3y=t^2y^{1/2}$, $y(1)=1$. Find the value of $y(2)$.
(a) 14 (b) 16 (c) 0 (d) 8

22. For $\lambda\in\mathbb{R}$ consider the system of differential equations.

$$x_1' = x_1+2x_2+2x_3,$$
$$x_2' = 2x_2+x_3,$$
$$x_3' = -x_3+2x_2+\lambda x_3$$

If $\vec{x}(t)=\vec{\alpha}te^{2t}$ (for some $\vec{\alpha}$) is a solution of the system then the value of λ is equal to :
(a) 2 (b) 4 (c) 6 (d) 1

23. Consider the Newton-Raphson method applied to approximate the square root to a positive number α. A recursion relation for the error $e_n=x_n-\sqrt{\alpha}$ is given by:

(a) $e_{n+1}=\frac{1}{2}\left(e_n+\frac{\alpha}{e_n}\right)$
(b) $e_{n+1}=\frac{1}{2}\left(e_n-\frac{\alpha}{e_n}\right)$
(c) $e_{n+1}=\frac{1}{2}\frac{e_n^2}{e_n+\sqrt{\alpha}}$
(d) $e_{n+1}=\frac{e_n^2}{e_n+2\sqrt{\alpha}}$

24. A body moves freely in a uniform gravitational field, the trajectory lies on which of the following curves in phase space ?
(a) Straight line in phase space.
(b) Parabola in phase space.
(c) Hyperbola in phase space.
(d) Ellipse in phase space.

25. Which of the following is a solution to $u_x+x^2u_y=0$ with $u(x,0)=e^x$?

(a) e^x (b) $e^{(x^3+y)^{1/3}}$ (c) $e^{(x^3-3y)^{1/3}}$ (d) $(x^2y+1)e^x$

26. Consider the differential equation:

$$x^2y''-2x(x+1)y'+2(x+1)y=0$$

If a polynomial is a solution then the degree of the polynomial is equal to:
(a) 1 (b) 2 (c) 3 (d) 4

27. Consider continuous solutions f of the following integral equation in [0, 1]

$$f^2(t)=1+2\int_0^t f(s)\,ds,\ \forall\, t\in[0,1]$$

Which of the following statements is true ?

(a) There is no solution. (b) There is exactly one solution.
(c) There are exactly two solutions. (d) There are more than two solutions.

28. The Euler equations satisfied by the extremals of the functional

$$J(y) = \int_0^5 \left[y^2 + x^3 y' \right] dx$$

define a solution curve in the *xy*-plane which is :

(a) linear (b) quadratic (c) cubic (d) trigonometric

29. Let $X_0 = 0$ and for $k \geq 1$, let X_k be a random variable with Binomial $\left(k, \frac{1}{2}\right)$ distribution. Let N be a Poisson random variable with mean 1. Assume that for every $k \geq 1$. X_k and N are independent and set $Y = X_N$. Given that $Y = 3$, what is the probability that $N = 3$?

(a) $\frac{1}{6}e^{-1}$ (b) $e^{-1/2}$ (c) $\frac{1}{6}e^{-1/2}$ (d) $\frac{1}{48}e^{-1/2}$

30. Let $X_1, X_2, \ldots$ be i.i.d. Normal random variables with mean 2 and variance 3. Let N be a Poisson random variable with mean 4 that is independent of $\{X_1, X_2, \ldots\}$. Let $Y = X_1 + \cdots + X_N$ if $N \geq 1$ and $Y = 0$ if $N = 0$. What is the variance of Y?

(a) 12 (b) 16 (c) 20 (d) 28

31. A sample of size n is to be drawn from a population of N households to estimate the proportion of the households which have more than one earning member. Let $0 < n < N$.

Sample-I is drawn using Simple Random Sampling without replacement.

Sample-II is drawn using Simple Random Sampling with replacement.

Let p_1 and p_2 denote the sample proportion in samples-I and II respectively. Let $\sigma_i^2 = \mathrm{Var}(p_i)$ for $i = 1, 2$. Which of the following statements is true ?

(a) p_1 is an unbiased estimate of the population proportion but p_2 is not.

(b) p_2 is an unbiased estimate of the population proportion but p_1 is not.

(c) $\sigma_1^2 < \sigma_2^2$

(d) $\sigma_2^2 < \sigma_1^2$

32. Suppose that X follows a distribution with probabilities density function $f_\theta(x) \propto x^{\theta-1}(1-x)^{\theta-1};\ 0 < x < 1$, $\theta > 0$. The uniformly most powerful critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$ based on a single observation is of the form.

(a) $(1-\alpha, 1]$ (b) $[0, \alpha)$ (c) $\left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right)$ (d) $\left[0, \frac{\alpha}{2}\right) \cup \left(1 - \frac{\alpha}{2}, 1\right]$

33. Consider a linear regression model of the form $y_i = \alpha + \beta x_i + \varepsilon_i$ based on the data $\{(x_i, y_i) : i = 1, 2, ..., 50\}$ (here ε_i are the error terms). Assume that not all x_i are the same. Which of the following is an admissible value of the leverage of the 11th observation ?

(a) 0 (b) 0.01 (c) 0.1 (d) 1.1

34. Let X be a uniform (0, 1) random variable. Suppose that given X, the random variable Y is uniform on $(0, X)$. Given X and Y, the random variable Z is uniform on $(Y, 1)$. What is the value of $E(Z)$?

(a) 1/8 (b) 3/8 (c) 5/8 (d) 1/2

35. Let $\left(N_t^1\right)$ and $\left(N_t^2\right)$ be two independent Poisson processes with intensities a, b respectively, with $a > b$. Let $M_t = \max\left\{N_t^1, N_t^2\right\}$ and $X_t = \max\left(N_t^1 - N_t^2, 0\right)$. Then which of the following is ***true*** ?

(a) X is a Poisson process with intensity $a - b$.

(b) X is a birth-and-death process with birth-rate a and death-rate b.

(c) M is a Poisson process with intensity max (a, b).

(d) M is a pure birth process with birth-rate $a + b$.

36. Consider the following Linear Programming Problem.

Maximize $4x + 5y$

subject to

$2x + 3y \le 14$

$x + 2y \le 9$

$x + y \le 6$

$x \ge 0,\ y \ge 0$

What is the optimal value of the objective function?

(a) 24 (b) 26 (c) 27 (d) 27

37. Let $X_1, X_2, ..., X_n$ be i.i.d. $N(\theta, 1)$ random variables. Suppose the prior distribution of θ is $N(0, \sigma^2)$.

Suppose we have squared error loss function. Let $\vec{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$. Then, the Bayes estimator of θ is:

(a) $\dfrac{\vec{X}}{1+\sigma^2}$ (b) $\vec{X}$ (c) $\dfrac{n\vec{X}\sigma^2}{1+n\sigma^2}$ (d) $\dfrac{n(\vec{X}+\sigma^2)}{1+n\sigma^2}$

38. Suppose that $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are independent and they have a common distribution which is uniform on the triangule with vertices (0, 0). $(\theta, 0)$ and $(0, \theta)$, where $\theta > 0$. A sufficient statistic for θ is :

(a) $\max_{1\le i\le n} X_i + \max_{1\le i\le n} Y_i$

(b) $\max_{1\le i\le n} (X_i + Y_i)$

(c) $\max_{1\le i\le n} |X_i + Y_i|$

(d) $\max\left\{\max_{1\le i\le n} X_i, \max_{1\le i\le n} Y_i\right\}$

39. Let X be a $p \times p$ matrix valued random variable having the Wishart distribution with parameters Σ (variance matrix) and m (degrees of freedom). Which of the following is necessarily true ?

[$A_{i,j}$ denotes the entry in i-th row and i-th column of a matrix A].

(a) $cX_{1,2}$ has Chi-square distribution with $m - p + 1$ degrees of freedom for a suitable constant c.

(b) $cX_{1,1}$ has Chi-square distribution with $m - p + 1$ degrees of freedom for a suitable constant c.

(c) $Y = AX$ has the Wishart distribution with variance matrix $A\Sigma A^T$ and degrees of freedom m for any non-singular $p \times p$ matrix A.

(d) $k \times k$ submatrix of X also has a Wishart distribution for $k < p$.

40. Let $X_1, X_2, ..., X_n$ be i.i.d. $N(\theta, \sigma^2)$ random variables where σ^2 is known. Let $\vec{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$. Then Minimum Variance Unbiased Estimator of $e^{2\theta}$ is given by:

(a) $\exp\left(2\bar{X} - \dfrac{\sigma^2}{n}\right)$ (b) $\exp\left(2\bar{X} - \dfrac{4\sigma^2}{n}\right)$ (c) $\exp\left(2\bar{X} - \dfrac{2\sigma^2}{n}\right)$ (d) $\exp\left(2n\bar{X} - \dfrac{2\sigma^2}{n}\right)$

PART - C (Mathematical Sciences)

1. Which of the following functions are uniformly continuous on (0, 1) ?

(a) $\dfrac{1}{x}$ (b) $\sin\dfrac{1}{x}$ (c) $x\sin\dfrac{1}{x}$ (d) $\dfrac{\sin x}{x}$

2. A 100 × 100 matrix $A = (a_{i,j})$ is such that $a_{i,j} = i$ if $i + j = 101$ and $a_{i,j} = 0$ otherwise. Which of the following statements are true about A ?

(a) A is similar to a diagonal matrix over $\mathbb{R}$. (b) A is not similar to a diagonal matrix over $\mathbb{C}$.

(c) One of the eigenvalues of A is 10. (d) None of the real eigenvalues of A exceeds 51.

3. Let A be a 3 × 3 nilpotent matrix. Which of the following statements are necessarily true ?

(a) $(I + A)^n = I$ for some $n > 0$ where I is the identity matrix.

(b) The column space of A is $\{0\}$.

(c) The eigenvalues of A are roots of 1.

(d) A^3 is diagonalizable.

4. Let $C_0(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ such that $\lim\limits_{x\to\pm\infty} f(x) = 0$. Let $C_0(\mathbb{R})$ be equipped with $\|\bullet\|$, the norm of uniform convergence. Let (f_n) be a sequence in $C_0(\mathbb{R})$ and $f \in C_0(\mathbb{R})$. Which of the following statements are ***correct*** ?

(a) If $f_n \to f$ uniformly on compact sets then $\|f_n - f\| \to 0$.

(b) If $\|f_n - f\| \to 0$ then $f_n \to f$ uniformly on compact sets.

(c) $f_n \to f$ uniformly on compact sets if and only if $\|f_n - f\| \to 0$.

(d) Neither of the two statements "$f_n \to f$ uniformly on compact sets" and "$\|f_n - f\| \to 0$" imply the other.

5. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Let V be the 3×3 matrix whose columns are v_1, v_2, v_3. Which of the following statements are necessarily true ?

(a) $VV^T = I$ (b) $V^T V = I$ (c) $V = V^T$ (d) Determinant of V is not zero

6. Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of natural numbers. Which of the following functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ are injective ?

(a) $f_1(m, n) = 2^m 3^n$ (b) $f_2(m, n) = mn + m + n$

(c) $f_3(m, n) = m^2 + n^3$ (d) $f_4(m, n) = m^2 n^3$

7. Let $W_1 = \left\{ \begin{bmatrix} a & b \\ -b & 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ and $W_2 = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ be two subspaces of $M_2(\mathbb{R})$. Which of the following statements are true ?

(a) $W_1 \cap W_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a proper subset of $W_1 \cap W_2$

(c) $W_1 + W_2 = M_2(\mathbb{R})$ (d) $W_1 + W_2$ is a proper subset of $M_2(\mathbb{R})$

8. Let $f(x, y) = (u(x, y), v(x, y)) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function. Let A denote the matrix of the derivative of f at the origin $(0, 0)$ with respect to the standard basis of $\mathbb{R}^2$. Assume $f(y, -x) = (v(x, y), -u(x, y))$ for all $(x, y) \in \mathbb{R}^2$. Which of the following statements are possibly true ?

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$ (d) $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

9. Let $f(x) = e^x$ for $x \in \mathbb{R}$. Which of the following statements are correct ?

(a) There is a real $C > 0$ such that $|f(x) - 1 - x| \le Cx^2$ for all $x \in \mathbb{R}$.

(b) There is a real $C > 0$ such that $\left| f(x) - 1 - x - \frac{x^2}{2} \right| \le C|x|^3$ for all $x \in [-1, 1]$.

(c) There is a real $C > 0$ such that $\left| f(x) - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!} \right| \le Cx^4$ for all $x \in \mathbb{R}$.

(d) There is a real $C > 0$ such that $\left| f(x) - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!} \right| \le Cx^4$ for all $x \in [-1, 1]$.

10. Let $f : [0, 1] \to (0, 1)$ be a function. Which of the following statements are False ?

(a) If f is onto, then f is continuous. (b) If f is continuous then f is not onto.

(c) If f is one-to-one, then f is continuous. (d) If f is continuous, then f is not one-to-one.

11. For any two non-negative integers, n, k define $f_{n,k}(x)$ on [0, 1] by:

$$f_{n,k}(x) = \begin{cases} x^n \sin(\pi/2x) - x^k & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

In which of the following cases is the function $f_{n,k}$, a function of bounded variation ?

(a) for all $n \geq 1$ and for all $k \geq 0$

(b) for all $n \geq 1$ and $k = 0$

(c) for all $n \geq 0$ and for all $k \geq 2$

(d) for all $n \geq 2$ and for all $k \geq 0$

12. Which of the following functions f admit an inverse in an open neighbourhood of the point $f(p)$?

(a) For $p = (1, 0)$ and $f(x, y) = (x^3 \exp y + y - 2x,\ 2xy + 2x)$

(b) For $p = (1, \pi)$ and $f(r, \theta) = (r\cos\theta, r\sin\theta)$

(c) For $p = 0$ and $f(x) = \begin{cases} x + 2x^2 \sin\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

(d) For $p = 0$ and $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

13. Let $M \in \mathbb{M}_n(\mathbb{R})$ with $M \neq 0, I_n$ but $M^2 = M$. Which of the following statements are true ?

(a) Null (M) is the eigenspace of M corresponding to the eigenvalue 0.

(b) Let $X \in \text{Col}(M)$ with $X \neq 0$. Then X is an eigenvector of M corresponding to the eigenvalue 1.

(c) Let $X \notin \text{Null}(M)$. Then X is an eigenvector of M corresponding to the eigenvalue 1.

(d) $\mathbb{R}^n = \text{Col}(M) + \text{Null}(M)$

14. Consider the identity functions $f(x) = x$ on $I := [0, 1]$. Let P_n be the partition that divides I into n equal parts. If $U(f, P_n)$ and $L(f, P_n)$ are the upper and lower Riemann sums, respectively, and $A_n = U(f, P_n) - L(f, P_n)$, then —

(a) $\lim_{n\to\infty} nA_n = 0$

(b) $\sum_{n=1}^{\infty} A_n$ is convergent

(c) A_n is strictly monotonically decreasing

(d) $\sum_{n=1}^{\infty} A_n A_{n+1} = 1$

15. Let $\{y_1, y_2, y_3, y_4\}$ be an orthonormal basis of $\mathbb{R}^4$. Which of the following are orthonormal bases ?

(a) $\{y_1 + y_2,\ y_1 - y_2,\ y_3,\ y_4\}$

(b) $\{y_3,\ y_4,\ -y_1,\ y_2\}$

(c) $\left\{\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}, y_3, y_4\right\}$

(d) $\left\{\frac{3y_1 + 4y_2}{5}, \frac{4y_1 - 3y_2}{5}, y_3, y_4\right\}$

16. Let A be an $m \times n$ matrix, let $A(1,:)$, $A(:,1)$ and $A(1,1)$ be the matrices obtained from A by deleting row 1 deleting column 1 and deleting both row 1 and column 1 respectively. Which of the following hold ?

(a) $(\text{rank } A) - 2 \leq \text{rank } A(1,1) \leq \text{rank } A$

(b) $\text{rank } A(1,:) = \text{rank } A(:,1)$

(c) $\text{rank } A = \text{rank } A(1,:) = \text{rank } A(:,1)$, then $\text{rank } A = \text{rank } A(1,1)$

(d) $\text{rank } A(1,:) + \text{rank } A(:,1) + 2 \geq 2 \text{ rank } A$

17. Let $\{x_n\}$ be a sequence of positive real numbers. Which of the following statements are true ?

(a) If the two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge, then the sequence $\{x_n\}$ converges.

(b) If $\left\{(-1)^n x_n\right\}$ converges, then the sequence $\{x_n\}$ converges.

(c) If $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n}$ exists then $\left\{(x_n)^{1/n}\right\}$ is bounded.

(d) If the sequence $\left\{x_n\right\}$ is unbounded then every subsequence is unbounded.

18. Let x, y be real numbers such that $0 < y \leq x$ and let n be a positive integer. Which of the following statements are true ?

(a) $ny^{n-1}(x-y) \leq x^n - y^n$

(b) $nx^{n-1}(x-y) \leq x^n - y^n$

(c) $ny^{n-1}(x-y) \geq x^n - y^n$

(d) $nx^{n-1}(x-y) \geq x^n - y^n$

19. Let $\mathbb{N} = \{1, 2, ...\}$ denote the set of possible integers. For $n \in \mathbb{N}$, let

$$A_n = \left\{(x, y, z) \in \mathbb{N}^3 : x^n + y^n = z^n \text{ and } z < n\right\}$$

Let $F(n)$ be the cardinality of the set A_n. Which of the following statements are true ?

(a) $F(n)$ is always finite for $n \geq 3$.

(b) $F(2) = \infty$

(c) $F(n) = 0$ for all n

(d) $F(n)$ is non-zero for some $n > 2$

20. Which of the following statements are true for $\alpha \in \mathbb{R}$?

(a) If α^3 is algebraic over $\mathbb{Q}$, then α is algebraic over $\mathbb{Q}$.

(b) α could be algebraic over $\mathbb{Q}\left[\sqrt{2}\right]$ but may not be algebraic over $\mathbb{Q}$.

(c) α need not be algebraic over any subfield of $\mathbb{R}$.

(d) There is an α which is not algebraic over $\mathbb{Q}\left[\sqrt{-1}\right]$.

21. For a positive integer n, let $\phi(n)$ be the Euler's ϕ-function. Which of the following statements are true for $n > 3$?

(a) $\phi(n)$ can never divide n.

(b) $\phi(n) \mid n \Rightarrow$ there exist integers x, y such that $nx + 6y = 1$

(c) $\phi(n) \mid n \Rightarrow n$ can have at most two distinct prime divisors.

(d) $\phi(n) \mid \phi(nk)$ for every positive integer k.

22. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc. Consider the family F of all holomorphic maps $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = \frac{1}{2}$ for $f \in F$, the possible values of $|f'(0)|$ are :

(a) $\frac{7}{8}$ (b) $\frac{5}{6}$ (c) $\frac{3}{4}$ (d) 1

23. Let $p > 2019$ be a prime number. Consider the polynomial

$f(x) = (x^2 - 3)(x^2 - 673)(x^2 - 2019)$

How many roots can f possibly have in the finite field $\mathbb{F}_p$?

(a) 0 (b) 2 (c) 3 (d) 6

24. Let G be the cyclic group of order 8 and $H = S_5$ be the permutation group of 5 elements. Which of the following statements are necessarily true ?

(a) There exists no non-trivial group homomorphism from G to H.
(b) There exists no injective group homomorphism from G to H.
(c) There exists no surjective group homomorphism from G to H.
(d) There are more than 20 different group homomorphisms from G to H.

25. Let $X = \mathbb{N} \cup \{\infty, -\infty\}$. Let τ be the topology on X consisting of subsets U of X such that either $U \subset \mathbb{N}$ or $X \setminus U$ is finite. Let $A = \mathbb{N} \cup \{\infty\}$ and $B = \mathbb{N} \cup \{-\infty\}$. Which of the following subsets are compact?

(a) A (b) $X \setminus A$ (c) $A \cup B$ (d) $A \cap B$

26. For $z \in \mathbb{C}$, let $\Re z$ denotes its real part. Let f be an entire function satisfying $|f(z)| \le |z|\,|\Re z|$ on $\mathbb{C}$. Which of the following statements are true ?

(a) $f(0) = 0$
(b) $f'(0) = 0$
(c) The only entire function satisfying the given property is $f(z) \equiv 0$.
(d) There exists a non-constant entire function satisfying the given property.

27. Fix a positive real number c. Consider the locus of all points $z \in \mathbb{C}$ such that $\left|\frac{z-i}{z+i}\right| = c$. Which of the following statements are true ?

(a) If $c > 1$, the locus is a circle centered on the imaginary axis.
(b) If $c < 1$, the locus is a circle centered on the real axis.
(c) If $c = 1$, the locus is a straight line parallel to the imaginary axis.
(d) If $c = 1$, the locus is a straight line not passing through the origin.

28. Which of the following statements are true ?
(a) Any compact topological space is metrizable.
(b) Any continuous image of a Hausdorff topological space is Hausdorff.
(c) If $f: X \to Y$ is continuous and one-to-one, and Y is Hausdorff, then X is Hausdorff.
(d) Intersection of two connected subsets of $\mathbb{R}$ with the usual topology is either empty or connected.

29. For positive integers m and n, let $\gcd(m, n)$ denote their greatest common divisor. Let $m > n$ be such that $\gcd(m, n) = 1$. Which of the following statements are true ?
(a) $\gcd(m-n, m+n) = 1$
(b) $\gcd(m-n, m+n)$ can have a prime divisor.
(c) There exists integers x, y such that $nx - my = 3$.
(d) $\gcd(m-n, m+n)$ can be an odd prime.

30. Consider the set $S := \{\exp(2\pi i\theta) : \theta \text{ is a rational number}\}$. For each $z \in S$ the set $\{z^{n!} : n$ is a positive integer$\}$ is :
(a) countable (b) countably infinite (c) uncountable (d) finite

31. Which of the following are solutions of the Laplace equation, $u_{xx} + u_{yy} = 0$ in the unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$?
(a) $x^5 + 2x^2y^3 - y^5$ (b) $x^2 + 2xy - y^2$
(c) $\cos(y)e^x + \sin(x)e^y$ (b) $\dfrac{1+x}{1+2x+x^2+y^2}$

32. Let x and y be continuously differentiable functions on $[0, \infty)$ that satisfy the respective initial value problems (ODEs)

$$\frac{dx}{dt} + (\sin t - 1)x = \log(1+t), \quad x(0) = 1;$$

$$\frac{dy}{dt} + (\sin t - 1)y = t, \quad y(0) = 2.$$

Define $z(t) = y(t) - x(t)$ for $t \geq 0$. Which of the following statements are true ?
(a) $z(1) \leq 1$ (b) $z(2) > z(1)$ (c) $z(1) > 1$ (d) $z(2) \leq z(1)$

33. Let $K(x, y)$ be a kernel in $[0, 1] \times [0, 1]$, defined as $K(x, y) = \sin(xy)$. The integral equation:

$$u(x) = \sin x + \int_0^1 K(x, y)u(y)\,dy$$

(a) is uniquely solvable and the solution is differentiable.
(b) has more than one differentiable solution.
(c) has no solution.
(d) is solvable and the solution is not differentiable.

34. Consider the numerical integration formula :

$$\int_{-1}^{1} g(x)\,dx \approx g(\alpha) + g(-\alpha),$$

where, $\alpha = (0.2)^{1/4}$. Which of the following statements are true ?

(a) The integration formula is exact for polynomials of the form $a + bx$ for all $a, b \in \mathbb{R}$.

(b) The integration formula is exact for polynomials of the form $a + bx + cx^2$, for all $a, b, c \in \mathbb{R}$.

(c) The integration formula is exact for polynomials of the form $a + bx + cx^2 + dx^3$, for all $a, b, c, d \in \mathbb{R}$.

(d) The integration formula is exact for polynomials of the form $a + bx + cx^3 + dx^4$ for all $a, b, c, d \in \mathbb{R}$.

35. Consider the partial differential equation (PDE)

$$x\left(\frac{\partial u}{\partial x}\right)^2 + y\left(\frac{\partial u}{\partial y}\right)^2 + (x+y)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + 1 = 0$$

Which of the following statements are True ?

(a) The general solution of the PDE can be expressed in the form:

$u(x, y) = ax + by + \dfrac{1}{a+b}$, where a and b are arbitrary constants.

(b) The general solution of the PDE can be expressed in the form:

$u(x, y) = f(ax + by) + \dfrac{1}{a+b}$, where a and b are arbitrary constants and f is an arbitrary function.

(c) The Charpit's equations are:

$$\frac{dx}{p^2 + pq} = \frac{dy}{q^2 + pq} = \frac{du}{p(p^2 + pq) + q(p^2 + pq)} = \frac{dp}{0} = \frac{dq}{0}$$

(d) The Charpit's equations are:

$$\frac{dx}{2px + (x+y)q - u} = \frac{dy}{2qy + (x+y)p - u}$$

$$= \frac{du}{p(2px + (x+y)q - u) + q(2qy + (x+y)p - u)} = \frac{dp}{0} = \frac{dq}{0}$$

36. Consider a body of unit mass moving under an attractive central force. At a certain radius R, the body moves in a circular orbit. Which of the following are true ?

(a) The force must be equal to $-\dfrac{L^2}{R^3}$, where L is the angular-momentum of the body.

(b) The force can be any strictly negative valued function of R.

(c) The force can only be of the inverse square law form.

(d) The force cannot be of the form $-kR$.

37. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz function. Consider the system of the ODEs given by:

$$\dot{x}_1 = \sin(e^{x_2}),\ \dot{x}_2 = f(x_1, x_2)$$

with initial condition $(x_1(0),\ x_2(0)) = (1, 1)$. Which of the following statements is true ?

(a) There is at most one local solution at time 0.

(b) There always exists a global solution defined in $[0, \infty)$.

(c) There might not be any solution around the time 0.

(d) There is at least one solution around time 0.

38. Let $f(x, y) = (y + x(1 - x^2 - y^2), -x + y(1 - x^2 - y^2))$ and consider the ODE $(\dot{x}, \dot{y}) = f(x, y)$ with initial condition $(x(0), y(0)) = \left(0, \frac{1}{2}\right)$. Which of the following statements are true ?

(a) $x^2(t) + y^2(t) \to \infty$ as $t \to \infty$

(b) $x^2(t) + y^2(t) \to 0$ as $t \to \infty$

(c) $x^2(t) + y^2(t)$ remains bounded as $t \to \infty$

(d) $x^2(t) + y^2(t) \to 1$ as $t \to \infty$

39. Let S be the set of all continuous functions $f:[0, 1] \to [0, \infty)$ that satisfy

$$\int_0^1 x^2 f(x)\,dx = \frac{1}{2}\int_0^1 x f^2(x)\,dx + \frac{1}{8}$$

Which of the following statements are true ?

(a) S is an empty set

(b) S has at most one element

(c) S has at least one element

(d) S has more than two elements

40. Let B be the unit ball in $\mathbb{R}^3$ centered at origin. The Euler-Lagrange equation corresponding to the functional $I(u) = \frac{1}{2}\int_B |\nabla u|^2\,dx - \frac{1}{5}\int_B |u|^5\,dx$ is

(a) $\Delta u = u^2$

(b) $\Delta u = -u^4$

(c) $\det(D^2 u) = u^4$

(d) $\Delta u = -|u|^3 u$

41. Let $K(x, y)$ be a kernel in $[0, 1] \times [0, 1]$ defined as:

$$K(x, y) = \begin{cases} x(1-y) & ; \quad 0 \le x \le y \le 1 \\ y(1-x) & ; \quad 0 \le y \le x \le 1; \end{cases}$$

and K be the associated integral operator. Then $K: L^2([0, 1]) \to L^2([0, 1])$

(a) is positive definite

(b) is self adjoint

(c) has a non-zero null space

(d) is onto

42. Consider the problem of extremising the functional $J(y) = \int_1^3 y(3x - y)\,dx$ with boundary conditions $y(1) = 1$ and $y(3) = 2$. Which of the following statements are true ?

(a) There is a unique extremal

(b) $y(x) = \frac{3x}{2}$ is an extremal

(c) $y(x) = \frac{x}{2} + \frac{1}{2}$ is an extremal

(d) There are no extremals.

60. Let A and B be two events. Which of the following are necessarily correct ?

(a) $P(A \cup B) \le P(A) + P(B)$

(b) $P(A \cup B) \ge P(A) + P(B)$

(c) $P(A \cap B) = P(A)\,P(B)$

(d) $P(A \cap B) \ge P(A) + P(B) - 1$

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(Answer Key) — PART - B (Mathematical Sciences)

1. (c)	**2.** (b)	**3.** (d)	**4.** (b)	**5.** (a)
6. (b)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (b)
11. (a)	**12.** (d)	**13.** (a)	**14.** (a)	**15.** (c)
16. (b)	**17.** (d)	**18.** (c)	**19.** (d)	**20.** (d)
21. (b)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (...)
26. (a)	**27.** (c)	**28.** (b)	**29.** ()	**30.** ()
31. ()	**32.** ()	**33.** ()	**34.** ()	**35.** ()
36. (b)	**37.** ()	**38.** ()	**39.** ()	**40.** ()

(Answer Key) — PART - C (Mathematical Sciences)

1. (c), (d)	**2.** (a), (c), (d)	**3.** (d)	**4.** (b)	**5.** (a,b,d)
6. (a)	**7.** (b), (d)	**8.** (a), (b), (d)	**9.** (b), (d)	**10.** (a,c,d)
11. (d)	**12.** (a), (b)	**13.** (a), (b), (d)	**14.** (c), (d)	**15.** (b), (d)
16. (a), (c), (d)	**17.** (b), (c)	**18.** (a), (d)	**19.** (a), (c)	**20.** (a,c,d)
21. (c), (d)	**22.** (c)	**23.** (b), (d)	**24.** (b), (c), (d)	**25.** (a,b,c)
26. (a), (b), (c)	**27.** (a)	**28.** (b), (d)	**29.** (b), (c)	**30.** (a), (d)
31. (b), (c), (d)	**32.** (b), (c)	**33.** (a)	**34.** (a), (d)	**35.** (a), (d)
36. (a), (b)	**37.** (a), (d)	**38.** (b), (c)	**39.** (a), (b)	**40.** (d)
41. (a), (b)	**42.** (d)	**43.** ()	**44.** ()	**45.** ()
46. ()	**47.** ()	**48.** ()	**49.** ()	**50.** ()
51. ()	**52.** ()	**53.** ()	**54.** ()	**55.** ()
56. ()	**57.** ()	**58.** ()	**59.** ()	**60.** (a), (d)