CSIR-NET-(PYQ) MATHEMATICAL SCIENCE
DECEMBER-2014-II

PART-B

21. Let $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. Then a necessary and sufficient condition for the sequence of polynomials $f_n(x) = b_n x + c_n x^2$ converge uniformly to 0 on the real line is

(a) $\lim\limits_{n\to\infty} b_n = 0$ and $\lim\limits_{n\to\infty} c_n = 0$

(b) $\sum_{n=1}^{\infty} = |b_n| < \infty$ and $\sum_{n=1}^{\infty} |c_n| < \infty$

(c) There exists a positive integer N such that $b_n = 0$ and $c_n = 0$ for all $n > N$

(d) $\lim\limits_{n\to\infty} c_n = 0$

Ans. **(c)**

22. Let k be a positive integer. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} z^n$ is

(a) k (b) k^{-k} (c) k^k (d) ∞

Ans. **(c)**

23. Suppose p is a polynomial with real coefficients. Then which of the following Statements is necessarily true?

(a) There is no root of the derivative p' between two real roots of the polynomial p

(b) There is exactly one root of the derivative p' between any two real roots of p

(c) There is exactly one root of the derivative p' between any two consecutive roots of p

(d) There is at least one root of the derivative p' between any two consecutive roots of p

Ans. **(d)**

24. Let $G = \{(x, f(x)) : 0 \le x \le 1\}$ be the graph of a real valued differentiable function f. Assume that $(1,0) \in G$. Suppose that the tangent vector to G at any point is perpendicular to the radius vector at that point. Then which of the following is true?

(a) G is the are of an ellipse (b) G is the arc of a circle

(c) G is a line segment (d) G is the are of a parabola

Ans. **(b)**

25. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ be a differentiable function such that $(Df)(x) = 0$ for all $x \in \Omega$. Then which of the following is true?

(a) f must be a constant function

(b) f must be constant on connected components of Ω

(c) $f(x) = 0$ or1 for $x \in \Omega$

(d) The range of the function f is a subset of $\mathbb{Z}$

Ans. **(b)**

26. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent. Let R be the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$. Then we can conclude that

(a) $0 < R < 1$ (b) $R = 1$ (c) $1 < R < \infty$ (d) $R = \infty$

Ans. **(b)**

27. Let A, B be $n \times n$ matrices such that $BA + B^2 = 1 - BA^2$ where I is then $n \times n$ identity matrix. Which of the following is always true ?

(a) A is nonsingular (b) B is nonsingular

(c) $A+B$ is nonsingular (d) AB is nonsingular

Ans. **(b)**

28. Which of the following matrices has the same row space as the matrix $\begin{pmatrix} 4 & 8 & 4 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix}$?

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Ans. **(a)**

29. The determinant of the $n \times n$ permutation matrix

$$\begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix}$$

(a) $(-1)^n$ (b) $(-1)^{\left[\frac{n}{2}\right]}$ (c) -1 (d) 1

Here $[x]$ denotes the greatest integer not exceeding x.

Ans. **(b)**

30. The determinant

$$\begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{vmatrix}$$

is equal to

(a) $(z-y)(z-x)(y-x)$ (b) $(x-y)(x-z)(y-z)$

(c) $(x-y)^2(y-z)^2(z-x)^2$ (d) $(x^2-y^2)(y^2-z^2)(z^2-x^2)$

Ans. **(a)**

31. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Ans. **(c)**

32. Let P be a 2×2 complex matrix such that $P^* P$ is the identity matrix, where P* is the"conjugate transpose of P. Then the"eigenvalues of P are

(a) real
(b) complex conjugates of each other
(c) reciprocals of each other
(d) of modulus 1

Ans. **(d)**

33. Let $p(z) = a_0 + a_1 z + ... + a_n z^n$ and $q(z) = b_1 z + b_2 z^2 + ... + b_n z^n$ be complex polynomials. If a_0, b_1 are non-zero complex numbers then the residue of $\dfrac{p(z)}{q(z)}$ at 0 is equal to

(a) $\dfrac{a_0}{b_1}$ (b) $\dfrac{b_1}{a_0}$ (c) $\dfrac{a_1}{b_1}$ (d) $\dfrac{a_0}{a_1}$

Ans. **(a)**

34. Let $\sum_0^\infty a_n z^n$ be a convergent power series such that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = R > 0$. Let p a be polynomial of degree d. Then the radius of convergence of the power series $\sum_{n=0}^\infty p(n) a_n z^n$ equals

(a) R (b) d (c) Rd (d) R+d

Ans. **(a)**

35. Let $\mathbb{Z}_+ = \{1,2, ...\}$ be the set of positive"integers. Let τ_1, := subspace topology on $\mathbb{Z}_+$ induced from the usual topology of $\mathbb{R}$,τ_2 := order topology on $\mathbb{Z}_+$ i.e., the topology with base

$\{\{x : 1 \le x < b\} : b \in \mathbb{Z}_+\} \cup \{\{x : a < x < b\} : a, b \in \mathbb{Z}_+\}$

τ_3 = discrete topology. Then

(a) $\tau_1 \ne \tau_3$ and $\tau_1 = \tau_2$ (b) $\tau_1 \ne \tau_2$ and $\tau_1 = \tau_3$
(c) $\tau_1 \ne \tau_3$ and $\tau_2 = \tau_3$ (d) $\tau_1 = \tau_2 = \tau_3$

Ans. **(d)**

36. The number of conjugacy classes in the permutation group S_6 is:

(a) 12 (b) 11 (c) 10 (d) 6

Ans. **(b)**

37. Find the degree of the field extension $\mathbb{Q}\left(\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2}\right)$ over $\mathbb{Q}$.

(a) 4 (b) 8 (c) 14 (d) 32

Ans. **(b)**

38. Let G be the Galois group of a field with nine elements over its subfield with three elements. Then the number of orbits for the action of G on the field with nine elements is

(a) 3 (b) 5 (c) 6 (d) 9

Ans. **(c)**

39. The number of surjective maps from a set of 4 elements to a set of 3 elements is

(a) 36 (b) 64 (c) 69 (d) 81

Ans. (a)

40. In the group of all invertible 4×4 matrices with entries in the field of 3 elements, any 3 Sylow subgroup has cardinality:

(a) 3 (b) 81 (c) 243 (d) 729

Ans. (d)

41. Let $P:\mathbb{R}\to\mathbb{R}$ be a polynomial of the form $P(x)=a_0+a_1x+a_2x^2$, with $a_0,a_1,a_2\in\mathbb{R}$ and $a_2\neq 0$

Let $E_1=\int_0^1 P(x)\,dx-\frac{1}{2}P\big(P(0)+P(1)\big)$, $E_2=\int_0^1 P(x)\,dx-P\left(\frac{1}{2}\right)$ If $|x|$ is the absolute value of $x\in\mathbb{R}$, then

(a) $|E_1|>|E_2|$ (b) $|E_2|>|E_1|$ (c) $|E_2|=|E_1|$ (d) $|E_2|=2|E_1|$

Ans. (a)

42. For $\lambda\in\mathbb{R}$, consider the boundary value problem

$$\left.\begin{aligned} x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\lambda y=0, \quad x\in[1,2] \\ y(1)=y(2)=0 \end{aligned}\right\}-(P_\lambda)$$

Which of the following statement is true ?

(a) |, There exists a $\lambda_0\in\mathbb{R}$ such that (P_λ) has a nontrivial solution for any $\lambda>\lambda_0$.

(b) $\{\lambda\in\mathbb{R}:(P_\lambda)$ has a nontrival solution$\}$ is a dense subset of $\mathbb{R}$.

(c) For any continuous function $f:[1,2]\to\mathbb{R}$ with $f(x)\neq 0$ for some $x\in[1,2]$, there exists a solution u of (P_λ) for some $\lambda\in\mathbb{R}$ such that $\int_1^2 fu\neq 0$

(d) There exists a $\lambda\in\mathbb{R}$ such that (P_λ) has two linearly independent solutions

Ans. ( )

43. Let $y:[0,\infty)\to\mathbb{R}$ be twice continuously differentiable and satisfy $y(x)+\int_0^x (x-s)u(s)\,ds=x^3/6$. Then

(a) $y(x)=\frac{1}{6}\int_0^x s^3\sin(x-s)\,ds$ (b) $y(x)=\frac{1}{6}\int_0^x s^3\cos(x-s)\,ds$

(c) $y(x)=\int_0^x s\sin(x-s)\,ds$ (d) $y(x)=\int_0^x s\cos(x-s)\,ds$

Ans. (c)

44. Consider the functional $J(y)=y^2(1)+\int_0^1 y'^2(x)\,dx$, $y(0)=1$ where $y\in C^2([0,1])$. If y extremizes f then

(a) $y(x)=1-\frac{1}{2}x^2$ (b) $y(x)=1-\frac{1}{2}x$

(c) $y(x)=1+\frac{1}{2}x$ (d) $y(x)=1+\frac{1}{2}x^2$

Ans. **(b)**

45. Let $u(x,t)=e^{i\omega x}v(t)$ with $v(0)=1$ be a solution to $\dfrac{\partial u}{\partial t}=\dfrac{\partial^3 u}{dx^3}$. Then

(a) $u(x,t)=e^{i\omega(x-\omega^2 t)}$ (b) $u(x,t)=e^{i\omega x-\omega^2 t}$

(c) $u(x,t)=e^{i\omega(x+\omega^2 t)}$ (d) $u(x,t)=e^{i\omega^3(x-t)}$

Ans. **(a)**

46. Let $y:\mathbb{R}\to\mathbb{R}$ be differentiable and satisfy the ODE:

$$\left.\begin{array}{l}\dfrac{dy}{dx}=f(y),\, x\in\mathbb{R}\\ y(0)=y(1)=0\end{array}\right\}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function. Then

(a) $y(x)=0$ if and only if $x\in\{0,1\}$ (b) y is bounded

(c) y is strictly increasing (d) $\dfrac{dy}{dx}$ is unbounded

Ans. **(b)**

47. The Charpit's equations for the PDE

$$up^2+q^2+x+y=0,\ p=\frac{\partial u}{\partial x},q=\frac{\partial u}{\partial y}$$

are given by

(a) $\dfrac{dx}{-1-p^3}=\dfrac{dy}{-1-qp^2}=\dfrac{du}{2p^2u+2q^2}=\dfrac{dp}{2pu}=\dfrac{dq}{2q}$

(b) $\dfrac{dx}{2pu}=\dfrac{dy}{2q}=\dfrac{du}{2p^2u+2q^2}=\dfrac{dp}{-1-p^3}=\dfrac{dq}{-1-qp^2}$

(c) $\dfrac{dx}{up^2}=\dfrac{dy}{q^2}=\dfrac{du}{0}=\dfrac{dp}{x}=\dfrac{dq}{y}$

(d) $\dfrac{dx}{2q}=\dfrac{dy}{2pu}=\dfrac{du}{x+y}=\dfrac{dp}{p^2}=\dfrac{dq}{qp^2}$

Ans. **(b)**

48. Consider a body of unit mass falling freely from rest under gravity with velocity *v*. If the air resistance retards the acceleration by *cv* where c is a constant, then

(a) $v=\dfrac{g}{c}\left[1+e^{ct}\right]$ (b) $v=\dfrac{g}{c}\left[1+e^{-ct}\right]$ (c) $v=\dfrac{g}{c}\left[1-e^{-ct}\right]$ (d) $v=\dfrac{g}{c}\left[1-e^{ct}\right]$

Ans. **(c)**

60. Consider the linear programming problem:

Minimize: $z = -2x - 5y$

subject to $3x + 4y \geq 5,\ x \geq 0,\ y \geq 0$

which of the following is correct ?

(a) Set of feasible solutions is empty.

(b) Set of feasible solutions is non-empty but there is no optimal solution.

(c) Optimal value is attained at $\left(0, \frac{5}{4}\right)$.

(d) Optimal value is attained at $\left(\frac{5}{3}, 0\right)$.

Ans. **(b)**

PART-C

61. Let E be a subset of $\mathbb{R}$. Then the characteristic function $\chi_E : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

(a) E is closed

(b) E is open

(c) E is both open and closed

(d) E is neither open nor closed

Ans. **(c)**

62. Suppose that P is a monic polynomial of degree n in one variable with real coefficients and K is a real number. Then which of the following statements is/are necessarily true ?

(a) If n is even and $K > 0$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = K\, e^{x_0}$

(b) If n is odd and $K < 0$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = K\, e^{x_0}$

(c) For any natural number n and $0 < K < 1$, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = K\, e^{x_0}$

(d) If n is odd and $K \in \mathbb{R}$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = K\, e^{x_0}$

Ans. **(a,b)**

63. Let $\{a_k\}$ be an unbounded, strictly increasing sequence of positive real numbers and $x_k = (a_{k+1} - a_k)/a_{k+1}$

Which of following statements is/are correc

(a) For all $n \geq m$, $\sum_{k=m}^{n} x_k > 1 - \frac{a_m}{a_n}$

(b) There exists $n \geq m$ such that $\sum_{k=m}^{n} x_k > \frac{1}{2}$

(c) $\sum_{k=1}^{\infty} x_k$ converges to a finite limit.

(d) $\sum_{k=1}^{\infty} x_k$ diverges to ∞

Ans. **(a,b,d)**

64. For a non-empty subset S and a point x in a connected metric space (X, d), let $d(x, S) = \inf\{d(x,y) : y \in S\}$. Which of the following statements is/are correct ?

(a) If S is closed and $d(x,S) > 0$ then x is not an accumulation point of S

(b) If S is open and $d(x,S) > 0$ then x is not an accumulation point of S

(c) If S is closed and $d(x,S) > 0$ then S does not contain x

(d) If S is open and $d(x,S) = 0$ then $x \in S$

Ans. **(a,b,d)**

65. Let f be a continuously differentiable function on $\mathbb{R}$. Suppose that

$$L = \lim_{x\to\infty}\left(f(x) + f'(x)\right)$$

exists. If $0 < L < \infty$, then which of the following statements is/are correct?

(a) If $\lim_{x\to\infty} f'(x)$ exists, then it is 0

(b) If $\lim_{x\to\infty} f(x)$ exists then it is L

(c) If $\lim_{x\to\infty} f'(x)$ exists then $\lim_{x\to\infty} f(x) = 0$

(d) If $\lim_{x\to\infty} f(x)$ exists then $\lim_{x\to\infty} f'(x) = L$

Ans. **(a, b)**

66. Let A be a subset of $\mathbb{R}$. Which of the following properties imply that A is compact ?

(a) Every continuous function f from A to $\mathbb{R}$ is bounded

(b) Every sequence $\{x_n\}$ in A has a convergent subsequence converging to a point in A

(c) There exists a continuous function from A onto [0,1]

(d) There is no one-one and continuous function from A onto (0,1)

Ans. **(a,b)**

67. Let f be a monotonically increasing function from [0,1] into [0,1]. Which of the following statements is/are true?

(a) f must be continuous at all but finitely many points in [0,1]

(b) f must be continuous at all but countably many points in [0,1]

(c) f must be Riemann integrable

(d) f must be Lebesgue integrable

Ans. **(b, c, d)**

68. Consider the normed linear spaces $X_1 = \left(C[0,1], \|\cdot\|_1\right)$ and $X_\infty = \left(C[0,1], \|\cdot\|_\infty\right)$, where $C[0,1]$ denotes the vector space of all continuous real valued functions on [0,1] and $\| f \|_1 = \int_0^1 |f(t)|\, dt$,

$\| f \|_\infty = \sup\left\{|f(t)| \mid t \in [0,1]\right\}$. Let U_1 and U_∞ be the open unit balls in X_1 and X_∞ respectively. Then

(a) U_∞ is a subset of U_1

(b) U_1 is a subset of U_∞

(c) U_∞ is equal to U_1

(d) Neither U_∞ is a subset of U_1 nor U_1 is a subset of U_∞

Ans. **(a)**

69. Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function. Let $G = \left\{(x, f(x)) : x \in X\right\}$ be the graph of f. Then

(a) G is homeomorphic to X

(b) G is homeomorphic to $\mathbb{R}$

(c) G is homeomorphic to $X \times \mathbb{R}$

(d) G is homeomorphic to $\mathbb{R} \times X$

Ans. **(a)**

70. Let A be a real $n\times n$ orthogonal matrix that is, $A^tA = AA^t = I_n$, the $n\times n$ identity matrix. Which of the following statements are necessarily true?

(a) $\langle Ax, Ay\rangle = \langle x, y\rangle\ \forall x, y \in \mathbb{R}^n$

(b) All eigenvalues of A are either +1 or –1

(c) The rows of A form an orthonormal basis of $\mathbb{R}^n$

(d) A is diagonalizable over $\mathbb{R}$

Ans. **(a,c)**

71. Which of the following matrices have Jordan canonical form equal to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$?

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Ans. (a,b,c)

72. Let A be a 3×4 and b be a 3×1 matrix with integer entries. Suppose that the system $Ax = b$ has a complex solution. Then

(a) $Ax = b$ has an integer solution.

(b) $Ax = b$ has a rational solution.

(c) The set of real solutions to $Ax = 0$ has a basis consisting of rational solutions.

(d) If $b \neq 0$ then A has positive rank

Ans. ( )

73. Let f be a non-zero symmetric bilinear form on $\mathbb{R}^3$. Suppose that there exist linear transformations $t_i : \mathbb{R}^3 \to \mathbb{R}, i = 1, 2$ such that for all $\alpha, \beta \in \mathbb{R}^3$, $f(\alpha, \beta) = T_1(\alpha) T_2(\beta)$. Then

(a) rank f =1

(b) $\dim \{\beta \in \mathbb{R}^3 : f(\alpha, \beta) = 0 \text{ for all } \alpha \in \mathbb{R}^3\} = 2$

(c) f is a positive semi-definite or negative semi-definite

(d) $\{\alpha : f(\alpha, \alpha) = 0\}$ is a linear subspace of dimension 2

Ans. (a,b,c,d)

74. The matrix $A = \begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix}$ satisfies:

(a) A is invertible and the inverse has all integer entries.

(b) det (A) is odd

(c) det (A) is divisible by 13.

(d) det (A) has at least two prime divisors

Ans. (c,d)

75. Let A be 5×5 matrix and let B be obtained by changing one element of A. Let r and s be the ranks of A and B respectively. Which of the following statements is/are correct ?

(a) $s \leq r + 1$ (b) $r - 1 \leq s$ (c) $s = r - 1$ (d) $s \neq r$

Ans. (a,b)

76. Let $M_n(K)$ denote the space of all $n \times n$ matrices with entries in a field K. Fix a non-singular matrix $A = (A_{ij}) \in M_n(K)$, and consider the linear map $T : M_n(K) \to M_n(K)$ given by: $T(X) = AX$ Then

(a) $\text{trace}(T) = n \sum_{i=1}^{n} A_{ii}$ (b) $\text{trace}(T) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}$

(c) rank of T is n^2 (d) T is non- singular

Ans. (a,c,d)

77. For arbitrary subspaces U, V and W of a finite dimensional vector space, which of the following hold:

(a) $U \cap (V+W) \subset U \cup V + U \cap W$ (b) $U \cap (V+W) \supset U \cap V + U \cap W$

(c) $(U \cap V) + W \subset (U+W) \cap (V+W)$ (d) $(U \cap V) + W \supset (U+W) \cap (V+W)$

Ans. **(b,c)**

78. Let A be a 4×7 real matrix and B be a 7×4 real matrix such that $AB = I_4$ where I_4 is the 4×4 identity matrix. Which of the following is/are always true ?

(a) rank (A) = 4

(b) rank (B) =7

(c) nullity (B) =0

(d) BA=I_7 where I_7 is the 7×7 identity matrix

Ans. **(a,c)**

79. Let f be an entire function on $\mathbb{C}$ and let Ω be a bounded open subset of $\mathbb{C}$.

Let $S = \{\operatorname{Re} f(z) + \operatorname{Im} f(z) \mid z \in \Omega\}$. Which of the following statements is/are necessarily correct ?

(a) S is an open set in $\mathbb{R}$ (b) S is a closed set in $\mathbb{R}$

(c) S is an open set of $\mathbb{C}$ (d) S is a discrete set in $\mathbb{R}$

Ans. **( )**

80. Let $u(x+iy) = x^3 — 3xy^2 + 2x$. For which of the following functions v, is $u + iv$ a holomorphic function on $\mathbb{C}$?

(a) $v(x+iy) = y^3 - 3x^2y + 2y$ (b) $v(x+iy) = 3x^2y - y^3 + 2y$

(c) $v(x+iy) = x^3 - 3xy^2 + 2x$ (d) $v(x+iy) = 0$

Ans. **(b)**

81. Let f be an entire function on $\mathbb{C}$. Let $g(z) = \overline{f(\bar{z})}$. Which of the following statements is/are correct ?

(a) if $f(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$ then $f = g$

(b) if $f(z) \in \mathbb{R}$ for all $z \in \{z \mid \operatorname{Im} z = 0\} \cup \{z \mid \operatorname{Im} z = a\}$ for some $a > 0$, then $f(z+ia) = f(z-ia)$ for all $z \in \mathbb{C}$

(c) if $f(z) \in \mathbb{R}$ for all $z \in \{z \mid \operatorname{Im} z = 0\} \cup \{z \mid \operatorname{Im} z = a\}$, for some $a > 0$, then $f(z+2ia) = f(z)$ for all $z \in \mathbb{C}$.

(d) if $f(z) \in \mathbb{R}$ for all $z \in \{z \mid \operatorname{Im} z = 0\} \cup \{z \mid \operatorname{Im} z = a\}$, for some $a > 0$, then $f(z+ia) = f(z)$ all $z \in \mathbb{C}$.

Ans. **(a,b,c)**

82. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and let r be a positive real number, Then

(a) $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le \sup_{|z|=r} |f(z)|^2$ (b) $\sup_{|z|=r} |f(z)|^2 \le \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$

(c) $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$ (d) $\sup_{|z|=r} |f(z)|^2 \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$

Ans. **(a,c)**

83. Let $X = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ be the unit circle inside $\mathbb{R}^2$. Let $f : X \to \mathbb{R}$ be a continuous function. Then:

(a) Image (f) is connected,

(b) Image (f) is compact.

(c) The given information is not sufficient to determine whether Image (f) is bounded.

(d) f is not injective

Ans. **(a,b,d)**

84. Let $\mathbb{R}[x]$ denote the vector space of all real polynomials. Let D: $\mathbb{R}[x]\to\mathbb{R}[x]$ denote the map $Df = \frac{df}{dx}, \forall f$. Then,

(a) D is one-one.

(b) D is onto.

(c) There exists E: $\mathbb{R}[x]\to\mathbb{R}[x]$ so that $D(E(f)) = f, \forall f$.

(d) There exists E: R[x] + R[x] so that $E(D(f)) = f, \forall f$.

Ans. **(b,c)**

85. Let G be a nonabelian group. Then, its order can be:

(a) 25 (b) 55 (c) 125 (d) 35

Ans. **(b,c)**

86. Which of the following are eigenvalues of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}?$$

(a) +1 (b) –1 (c) $+i$ (d) $-i$

Ans. **(a, b)**

87. Let $\mathbb{R}[x]$ be the polynomial ring over $\mathbb{R}$ in one variable. Let $I \subseteq \mathbb{R}[x]$ be an ideal. Then

(a) I is a maximal ideal if and only if I is a non-zero prime ideal

(b) I is a maximal ideal if and only if the quotient ring $\mathbb{R}[x]$/ I is isomorphic to $\mathbb{R}$

(c) I is a maximal ideal if and only if $I = (f(x))$ where $f(x)$ is a non-constant irreducible polynomial over $\mathbb{R}$

(d) I is a maximal ideal if and only if there exists a nonconstant polynomial $f(x) \in I$ of degree ≤ 2

Ans. **(b,c)**

88. Let G be a group of order 45, Then

(a) G has an element of order 9 (b) G has a subgroup of order 9

(c) G has a normal subgroup of order 9 (d) G has a normal subgroup of order 5

Ans. **(b,c,d)**

89. Which of the following is/are true?

(a) Given any positive integer n, there exists a field extension of $\mathbb{Q}$ of degree n.

(b) Given a positive integer n, there exist fields F and K such that $F \subseteq K$ and K is Galois over F with $[K : F] = n$,

(c) Let K be a Galois extension of $\mathbb{Q}$ with $[K : \mathbb{Q}] = 4$. Then there is a field L such that

$K \supseteq L \supseteq \mathbb{Q}, [L : \mathbb{Q}] = 2$ and L is a Galois extension of $\mathbb{Q}$.

(d) There is an algebraic extension K of $\mathbb{Q}$ such that $[K: \mathbb{Q}]$ is not finite.

Ans. (a,b, c,d)

90. Let $A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, where $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$, Then we must have:

(a) For any $n \geq 1$, $A^n = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $x = \cos(\theta / n), y = \sin(\theta / n)$

(b) $tr(A) \neq 0$

(c) $A^t = A^{-1}$

(d) A is similar to a diagonal matrix over $\mathbb{C}$

Ans. (a, c, d)

91. The system of ODE

$$\left.\begin{aligned} \frac{dx}{dt} &= (1+x^2)y \quad , t \in \mathbb{R} \\ \frac{dy}{dt} &= -(1+x^2)x \quad t \in \mathbb{R} \end{aligned}\right\}$$
$$(x(0), y(0)) = (a,b)$$

has a solution:

(a) only if $(a,b) = (0,0)$

(b) for any $(a,b) \in \mathbb{R} \times \mathbb{R}$

(c) such that $x^2(t) + y^2(t) = a^2 + b^2$ for all $t \in \mathbb{R}$

(d) such that $x^2(t) + y^2(t) \to \infty$ as $t \to \infty$ if $a > 0$ and $b > 0$

Ans. (b,c)

92. Let y: $\mathbb{R} \to \mathbb{R}$ be a solution of the ODE

$$\left.\begin{aligned} &\frac{d^2y}{dx^2} - y = e^{-x}, x \in \mathbb{R} \\ &y(0) = \frac{dy}{dx}(0) = 0 \end{aligned}\right\}$$

then

(a) y attains its minimum on $\mathbb{R}$

(b) y is bounded on $\mathbb{R}$

(c) $\lim_{x \to \infty} e^{-x} y(x) = \frac{1}{4}$

(d) $\lim_{x \to \infty} e^{x} y(x) = \frac{1}{4}$

Ans. (a,c)

93. Let $u \in c^2([0,1])$ satisfy for some $\lambda \neq 0$ and $a \neq 0$

$$u(x)+\frac{\lambda}{2}\int_0^1 |x-s|\,u(s)\,ds = ax+b$$ .Then u also satisfies

(a) $\frac{d^2u}{dx^2}+\lambda u=0$

(b) $\frac{d^2u}{dx^2}-\lambda u=0$

(c) $\frac{du}{dx}-\frac{\lambda}{2}\int_0^1 \frac{x-s}{|x-s|}u(s)\,ds=a$

(d) $\frac{du}{dx}+\frac{\lambda}{2}\int_0^1 \frac{x-s}{|x-s|}u(s)\,ds=a$

Ans. **(a,d)**

94. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function with non vanishing derivative. The Newton's method for finding a root of $f(x)=0$ is the same as

(a) fixed point iteration for the map $g(x)=x-f(x)/f'(x)$

(b) Forward Euler method with unit step length for the differential equation $\frac{dy}{dt}+\frac{f(y)}{f'(y)}=0$

(c) fixed point iteration for $g(x)=x+f(x)$

(d) fixed point iteration for $g(x)=x-f(x)$

Ans. **(a,b)**

95. Consider a particle moving with coordinates $(x(t), y(t))$ on a smooth curve $\phi(x,y)=0$. If the particle moves from $(x(0), y(0))$ to $(x(\tau), y(\tau))$ for $\tau > 0$ such that its kinetic energy is minimized, then

(a) $\frac{\ddot{x}}{\phi_x}=\frac{\ddot{y}}{\phi_y}$

(b) $\dot{x}^2(0)+\dot{y}^2(0)=\dot{x}^2(\tau)+\dot{y}^2(\tau)$

(c) $\dot{x}\phi_x+\dot{y}\phi_y=(0)$

(d) $\dot{x}^2(0)=\dot{x}^2(\tau)$

Ans. **(a,c)**

96. Let $y \in C^2([0,\pi])$ satisfying $y(0)=y(\pi)=0$ and $\int_0^\pi y^2(x)\,dx=1$ extremize the functional

$$J(y)=\int_0^\pi \left(y'^{(x)}\right)^2 dx;\ y'=\frac{dy}{dx}$$. Then

(a) $y(x)=\sqrt{\frac{2}{\pi}}\sin x$

(b) $y(x)=-\sqrt{\frac{2}{\pi}}\sin x$

(c) $y(x)=\sqrt{\frac{2}{\pi}}\cos x$

(d) $y(x)=-\sqrt{\frac{2}{\pi}}\cos x$

Ans. **(a, b)**

97. Consider the Cauchy problem of finding $u = u(x,t)$ such that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \text{ for } x \in \mathbb{R}, t > 0$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}$$

Which choice(s) of the following functions for u_0 yield a C^1 solution $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$.

(a) $u_0(x) = \frac{1}{1+x^2}$ (b) $u_0(x) = x$

(c) $u_0(x) = 1 + x^2$ (d) $u_0(x) = 1 + 2x$

Ans. **(b,d)**

98. Let P, Q be continuous real valued functions defined on $[-1, 1]$ and $u_i : [-1,1] \to \mathbb{R}$, $i = 1, 2$ be solutions of the ODE $\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0$, $x \in [-1,1]$ satisfying $u_1 \geq 0, u_2 \leq 0$ and $u_1(0) = u_2(0) = 0$.

Let w denote the Wronskian of u_1 and u_2 then

(a) u_1 and u_2 are linearly independent (b) u_1 and u_2 are linearly dependent

(c) $w(x) = 0$ for all $x \in [-1,1]$ (d) $w(x) \neq 0$ for some $x \in [-1,1]$

Ans. **(b,c)**

99. Which of the following approximations for estimating the derivative of a smooth function f at a point x is of order 2 (i.e. the error term is $o(h^2)$)

(a) $f'(x) \approx \frac{f(x+h) - f(x)}{h}$ (b) $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$

(c) $f'(x) \approx \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h}$ (d) $f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$

Ans. **(b,c,d)**

100. Let $u(x, t)$ satisfy for $x \in \mathbb{R}, t > 0$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + 2\frac{\partial^2 u}{\partial x^2} = 0$$

A solution of the form $u = e^{ix}v(t)$ with $v(0) = 0$ and $v'(0) = 1$

(a) is necessarily bounded (b) satisfies $|u(x,t)| < e^t$

(c) is necessarily unbounded (d) is oscillatory in x.

Ans. **(b,c,d)**

101. Let $u = u(x,t)$ be the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 1 \qquad x \in \mathbb{R}, t > 0$$

$$u(x,0) = -x^2 \qquad x \in \mathbb{R}$$

then

(a) $u(x,t)$ exists for all $x \in \mathbb{R}$ and $t > 0$. (b) $|u(x,t)| \to \infty$ $t \to t^*$ for some $t^* > 0$ and $x \neq 0$

(c) $u(x,t) \leq 0$ for all $x \in \mathbb{R}$ and for all $t < 1/4$. (d) $u(x,t) > 0$ for some $x \in \mathbb{R}$ and $0 < t < 1/4$.

Ans. **(b, d)**

102. Let $y(t)$ satisfy the differential equation $y' = \lambda y; y(0) = 1$. Then the backward Euler method, for $n \geq 1$ and $h > 0$

$\dfrac{y_n - y_{n-1}}{h} = \lambda y_n;\ y_0 = 1$ yields

(a) a first order approximation to $e^{\lambda nh}$

(b) a polynomial approximation to $e^{\lambda nh}$

(c) a rational function approximation to $e^{\lambda nh}$

(d) a Chebyshev polynomial approximation to $e^{\lambda nh}$

Ans. **(a,c)**