CSIR-NET-(P YQ) MATHEMATICAL SCIENCE
DECEMBER-2017-II

PART-B

1. Let $A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\in M_2(\mathbb{R})$ and $\phi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be the bilinear map defined by $\phi(v,w)=v^T Aw$. Choose the correct statement from below:

(a) $\phi(v,w)=\phi(w,v)$ for all $v,w\in\mathbb{R}^2$

(b) there exists nonzero $v\in\mathbb{R}^2$ such that $\phi(v,w)=0$ for all $w\in\mathbb{R}^2$

(c) there exists a 2×2 symmetric matrix B such that $\phi(v,v)=v^T\ Bv$ for all $v\in\mathbb{R}^2$

(d) the $\psi:\mathbb{R}^4\to\mathbb{R}$ map defined by $\psi\left(\begin{bmatrix}v_1\\ v_2\\ w_1\\ w_2\end{bmatrix}\right)=\phi\left(\begin{bmatrix}v_1\\ v_2\end{bmatrix},\begin{bmatrix}w_1\\ w_2\end{bmatrix}\right)$ is linear

Ans. (b)

2. Let $A=\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & 1\\ 2 & 3 & \alpha\end{bmatrix}$ and $b=\begin{bmatrix}1\\ 3\\ \beta\end{bmatrix}$. Then the system $AX=b$ over the real numbers has

(a) no solution whenever $\beta\neq 7$

(b) an infinite number of solutions whenever $\alpha\neq 2$

(c) an infinite number of solutions if $\alpha=2$ and $\beta\neq 7$

(d) a unique solution if $\alpha\neq 2$

Ans. (c)

3. Let $A=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$. Then the smallest positive integer n such that $A^n=\mathrm{I}$ is

(a) 1 (b) 2 (c) 4 (d) 6

Ans. (b)

4. Let A be a real symmetric matrix and $B=I+iA$, where $i^2=-1$. Then

(a) B is invertible if and only if A is invertible

(b) all eigenvalues of are B necessarily real

(c) $B-I$ is necessarily invertible

(d) B is necessarily invertible

Ans. (b)

5. Let $S=\{x\in[-1,4]\,|\,\sin(x)>0\}$. Which of the following is true?

(a) inf (S) < 0 (b) sup (S) does not exist

(c) sup (S) = π (d) inf (S) = $\pi/2$

Ans. **(c)**

6. Let k be a positive integer and let $\mathbf{S_k}=\{x\in[0,1]\,|$ a decimal expansion of x has a prime digit at its k^{th} place$\}$.

Then the Lebesgue measure of S_k is

(a) 0 (B) 4/10 (c) $(4/10)^k$ (d) 1

Ans. **(b)**

7. Which of the following is necessarily true for a function $f:X\to Y$?

(a) if f is injective, then there exists $g:Y\to X$ such that $f(g(y)=y$ for all $y\in Y$

(b) if f is surjective, then there exists $g:Y\to X$ such that $f(g(y))=y$ for all $y\in y$

(c) if f is injective and Y is countable then X is finite

(d) if f is surjective and X is uncountable then Y is countably infinite

Ans. **(b)**

8. Let $S=\{f:\mathbb{R}\to\mathbb{R}\,|\,\exists\in>0$ such that $\forall\delta>0.\,|x-y|<\delta\Rightarrow|f(x)-f(y)|<\in\}$. Then

(a) S = {f: $\mathbb{R}\to\mathbb{R}\,|\,f$ is continuous}

(b) S = {f: $\mathbb{R}\to\mathbb{R}\,|\,f$ is uniformly continuous}

(c) S = {f: $\mathbb{R}\to\mathbb{R}\,|\,f$ is bounded}

(d) S = {f: $\mathbb{R}\to\mathbb{R}\,|\,f$ is constant}

Ans. **(c)**

9. Let $f:(0,\infty)\to\mathbb{R}$ be uniformly continuous.

Then

(a) $\lim\limits_{x\to0^+}f(x)$ and $\lim\limits_{x\to\infty}f(x)$ exist

(b) $\lim\limits_{x\to0^+}f(x)$ exist but $\lim\limits_{x\to\infty}f(x)$ need not exist

(c) $\lim\limits_{x\to0^+}f(x)$ need not exist but $\lim\limits_{x\to\infty}f(x)$ exists

(d) neither $\lim\limits_{x\to0^+}f(x)$ nor $\lim\limits_{x\to\infty}f(x)$ need exist

Ans. **(d)**

10. Let D be a subset of the real line. Consider the assertion: "Every infinite sequence in D has a subsequence which converges in D". This assertion is true if

(a) $D=[0,\infty)$ (b) $D=[0,1]\cup[3,4]$ (c) $D=[-1,1)\cup(1,2]$ (d) $D=(-1,1]$

Ans. **(d)**

11. Let $\{a_n\}_{n\ge1}$ be a sequence of real numbers satisfying $a_1\ge1$ and $a_{n+1}\ge a_n+1$ for all $n\ge1$. Then which of the following is necessarily true ?

(a) The series $\sum\limits_{n=1}^{\infty}\frac{1}{a_n^2}$ diverges (b) The sequence $\{a_n\}_{n\ge1}$ is bounded

(c) The series $\sum\limits_{n=1}^{\infty}\frac{1}{a_n^2}$ converges (d) The series $\sum\limits_{n=1}^{\infty}\frac{1}{a_n}$ converges

Ans. **(d)**

12. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}_{\geq 0}$ denote the set $\{0, 1, 2, 3, ...\}$. Consider the map $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(m,n) = 2^m \cdot (2n+1)$. Then the map f is

(a) onto (surjective) but not one–one (injective)
(b) one–one (injective) but not onto (surjective)
(c) both one–one and onto
(d) neither one–one nor onto

Ans. **(c)**

13. Let A be a connected open subset of $\mathbb{R}^2$. The number of continuous surjective functions from $\bar{A}$ (the closure of A in $\mathbb{R}^2$) to $\mathbb{Q}$ is:

(a) 1 (b) 0 (c) 2 (d) not finite

Ans. **(d)**

14. Let R be a subring of $\mathbb{Q}$ containing 1. Then which of the following is necessarily true?

(a) R is a principal ideal domain (PID)
(b) R contains infinitely many prime ideals
(c) R contains a prime ideal which is not a maximal ideal
(d) for every maximal ideal m in R, the residue field R/m is finite

Ans. **(c)**

15. The group S_3 of permutations of $\{1,2,3\}$ acts on the three dimensional vector space over the finite field $\mathbb{F}_3$ of three elements, by permuting the vectors in basis $\{e_1, e_2, e_3\}$ by $\sigma \cdot e_i = e_{\sigma(i)}$, for all $\sigma \in S_3$. The cardinality of the set of vectors fixed under the above action is

(a) 0 (b) 3 (c) 9 (d) 27

Ans. **(d)**

16. Let $f : \mathbb{Z} \to (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ be the function $f(n)$= (n mod 4, n mod 6). Then

(a) (0 mod 4, 3 mod 6) is in the image of f
(b) (a mod 4, b mod 6) is in the image of f, for all even integers a and b
(c) image of f has exactly 6 elements
(d) kernel of f = $24\mathbb{Z}$

Ans. **(c)**

17. Let $\mathbb{D}$ be the open unit disc in the complex plane and $U = \mathbb{D} \setminus \left\{-\frac{1}{2}, \frac{1}{2}\right\}$. Also, let $H_1 = \{ f : \mathbb{D} \to \mathbb{C} \mid f$ holomorphic and bounded$\}$ and $H_2 = \{ f : U \to \mathbb{C} \mid f$ is holomorphic and bounded$\}$.
Then the map $r : H_1 \to H_2$ given by $r(f) = f|_U$, the restriction of f to U, is

(a) injective but not surjective
(b) surjective but not injective
(c) injective and surjective
(d) neither injective nor surjective

Ans. **(b)**

18. Let C be the circle of radius 2 with centre at the origin in the complex plane, oriented in the anti-clockwise direction. Then the integral $\oint_C \frac{dz}{(z-1)^2}$ is equal to

(a) $\frac{1}{2\pi i}$ (b) $2\pi i$ (c) 1 (d) 0

Ans. **(b)**

19. Let f be a holomorphic function in the open unit disc such that $\lim_{z\to 1} f(z)$ does not exist. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of f about $z = 0$ and let R be its radius of convergence. Then

(a) R = 0 (b) 0 < R <1 (c) R = 1 (d) R > 1

Ans. **(a)**

20. The function $f:\mathbb{C}\to\mathbb{C}$ defined by $f(z)=e^z+e^{-z}$ has

(a) finitely many zeros (b) no zeros

(c) only real zeros (d) has infinitely many zeros

Ans. **(b)**

21. Let $I(m)$, denote the moment of inertia of a regular solid tetrahedron about an axis m passing through its centre of gravity. Which of the following is true ?

(a) if the axis ℓ passes through a vertex and the axis ℓ' does not pass through a vertex then $I(\ell) > I(\ell)$

(b) if the axis ℓ passes through the mid-point of an edge and ℓ' is any other axis then $I(\ell) > I(\ell')$

(c) I (ℓ) is the same for all axes ℓ

(d) if the axis ℓ passes through a vertex and the axis ℓ' does not pass through a vertex then $I(\ell) < I(\ell')$

Ans. **(c)**

22. Let $u(x,t)$ be a solution of the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ in a rectangle $[0,\pi]\times[0,T]$ subject to the boundary conditions $u(0,t)=u(\pi,t)=0$, $0\le t\le T$ and the initial condition $u(x,0)=\varphi(x)$, $0\le x\le\pi$.

If $f(x)=u(x,T)$, then which of the following is true for a suitable kernel $k(x,y)$?

(a) $\int_0^{\pi} k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$

(b) $\varphi(x)+\int_0^{\pi} k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$

(c) $\int_0^{\pi} k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$

(d) $\varphi(x)+\int_0^{x} k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$

Ans. **(c)**

23. Let $X=\{u\in C^1[0,1]\,|\,u(0)=u(1)=0\}$ and define $J:X\to\mathbb{R}$ by $J(u)=\int_0^1 e^{-u'(x)^2}\,dx$

Then

(a) J does not attain its infimum

(b) J attains its infimum at a unique $u\in X$

(c) J attains its infimum at exactly two elements $u\in X$

(d) J attains its infimum at infinitely many $u\in X$

Ans. **(c)**

24. The iterative method $x_{n+1} = g(x_n)$ for the solution of $x^2 - x - 2 = 0$ converges quadratically in a neighbourhood of the root $x = 2$ if $g(x)$ equals

(a) $x^2 - 2$ (b) $(x-2)^2 - 6$ (c) $1 + \frac{2}{x}$ (d) $\frac{x^2+2}{2x-1}$

Ans. **(a)**

25. Let $u(x,t)$ be the solution of the initial value problem

$u_{tt} - u_{xx} = 0$

$u(x,0) = x^3$

$u_t(x,0) = \sin x$

Then $u(\pi,\pi)$ is

(a) $4\pi^3$ (b) π^3 (c) 0 (d) 4

Ans. **(d)**

26. The set of real numbers λ for which the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0$$

has nontrivial solutions is

(a) $(-\infty, 0)$ (b) $\{\sqrt{n} \mid n \text{ is a possitive integer}\}$

(c) $\{n^2 \mid n \text{ is a possitive integer}\}$ (d) $\mathbb{R}$

Ans. **(a)**

27. Let D denote the unit disc given by $\{(x,y) \mid x^2 + y^2 \le 1\}$ and let D^C be its complement in the plane. The partial differential equation

$$(x^2-1)\frac{\partial^2 u}{\partial x^2} + 2y\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

(a) parabolic for all $(x,y) \in D^C$ (b) hyperbolic for all $(x,y) \in D$

(c) hyperbolic for all $(x,y) \in D^C$ (d) parabolic for all $(x,y) \in D$

Ans. **(a)**

28. Consider the differential equation $(x-1)y'' + xy' + \frac{1}{x}y = 0$ Then

(a) $x = 1$ is the only singular point

(b) $x = 0$ is the only singular point

(c) both $x = 0$ and $x = 1$ are singular points

(d) neither $x = 0$ nor $x = 1$ are singular points

Ans. **(c)**

29. A parallel system consists of n identical components. The lifetimes of the components are independent identically distributed uniform random variables with mean 30 hours and range 60 hours. If the expected lifetime of the system is 50 hours, then the value of n is

(a) 3 (b) 4 (c) 5 (d) 6

Ans. (a)

30. Let X and Y be independent exponential random variables. If $E[X]=1$ and $E[Y]=\frac{1}{2}$ then $P(X>2Y \mid X>Y)$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Ans. (c)

31. Suppose we draw a random sample of size n from a population of size N, where $1<n<N$, using simple random sampling without replacement scheme. Let P be the population proportion of units possessing a particular attribute and p be the corresponding sample proportion. Which of the following is an unbiased estimator for $P(1-P)$?

(a) $p(1-p)$ (b) $\frac{N-n}{N-1}(1-p)$ (c) $\frac{n(N-1)}{N(n-1)}p(1-p)$ (d) $\frac{N(n-1)}{n(N-1)}p(1-p)$

Ans. (b)

32. Suppose $X_1 \sim N_{p_1}(0,\Sigma_1), X_2 \sim N_{p_2}(0,\Sigma_2)$, where X_1 and X_2 are independently distributed. If $p_1>p_2$ and Σ_1,Σ_2 are positive definite then which of the following statements is necessarily true?

(a) $X_1^T\Sigma_1 X_1 + X_2^T\Sigma_2 X_2 \sim \chi^2_{p_1+p_2}$

(b) $X_1^T\Sigma_1^{-1} X_1 + X_2^T\Sigma_2^{-1} X_2 \sim \chi^2_{p_1+p_2}$

(c) $X_1^T\Sigma_1^{-1} X_1 - X_2^T\Sigma_2^{-1} X_2 \sim \chi^2_{p_1-p_2}$

(d) $\frac{p_1 X_1^T\Sigma_1^{-1} X_1}{p_2 X_2^T\Sigma_2^{-1} X_2} \sim F_{p_1,p_2}$

Ans. (d)

33. Consider the following regression problem

$Y_i = \alpha + \beta i + \in_i;\ i=1,.....,n$

Here $\in_i, i=1,2,...,n$ are i.i.d N(0, 1) random variables. It is assumed that $\alpha \neq 0$ and β is known. If $\hat{\alpha}_n$ is the MLE of α, which of the following statements is true ?

(a) $\lim_{n\to\infty} E(\hat{\alpha}_n) \neq \alpha$

(b) $\lim_{n\to\infty} E(\hat{\alpha}_n) = 0$

(c) $\lim_{n\to\infty} \text{Var}(\hat{\alpha}_n) = \infty$

(d) $\lim_{n\to\infty} \text{Var}(\hat{\alpha}_n) = 0$

Ans. (a)

34. Let $X_1, X_2,.....$ be a random sample from uniform $(0, 3\theta), \theta>0$. Define $T_n = \frac{1}{3}\max\{X_1, X_2, ...X_n\}$. Which of the following is NOT true ?

(a) T_n is consistent for θ

(b) T_n is unbiased for θ

(c) T_n is a sufficient statistic

(d) T_n is complete

Ans. (b)

35. Let X be a random sample of size 1 from a Cauchy distribution with probability density function

$$f_\theta(x)=\frac{1}{\pi}\left(\frac{1}{1+(x-\theta)^2}\right), -\infty<x<\infty, \text{ where } \theta\in(-\infty,\infty)$$. For testing $H_0:\theta=-1$ against $H_1:\theta=0$, the following test is suggested. Reject H_0 if $\frac{X}{\sqrt{1+X^2}}>C$, otherwise do not reject H_0. What is the value of C so that the power of the test is 0.5 ?

(a) $\frac{\pi}{4}$ (b) 0

(c) $\tan^{-1}\left(\frac{1}{2}\right)$ (d) a solution of $\tan^{-1}\sqrt{\frac{c}{1-c}}=\frac{\pi}{3}$

Ans. (b)

36. Let X_1 and X_2 be a random sample of size two from a distribution with probability density function

$$f_\theta(x)=\theta\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}+(1-\theta)\frac{1}{2}e^{-|x|}$$

$-\infty<x<\infty$,

where $\theta\in\left\{0,\frac{1}{2},1\right\}$. If the observed values of X_1 and X_2 are 0 and 2, respectively, then the maximum likelihood estimate of θ is

(a) 0 (b) 1/2 (c) 1 (d) not unique

Ans. (d)

37. X, Y are independent exponential random variables with means 4 and 5, respectively. Which of the following statements is true ?

(a) $X+Y$ is exponential with mean 9 (b) XY is exponential with mean 20

(c) max (X,Y) is exponential (d) min (X, Y) is exponential

Ans. (b)

38. Consider a Markov chain $\{X_n \mid n\geq 0\}$ with state space {1, 2, 3}and transition matrix

$$P\begin{pmatrix}0 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/2 & 0\end{pmatrix}$$ then $P(X_3=1\mid X_0=1)$ equals

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

Ans. (c)

39. Let $\psi(t)=e^{-|t|-\frac{t^2}{2}}$ and $\varphi(t)=\frac{e^{-|t|+e^{-\frac{t^2}{2}}}}{2}$. Which of the following is true?

(a) ψ is a characteristic function but φ is not (b) φ is a characteristic function but ψ is not

(c) both ψ and φ are characteristic functions (d) neither ψ nor φ is a characteristic function

Ans. (d)

40. There are five empty boxes. Balls are placed independently one after another in randomly selected boxes. The probability that the fourth ball is the first to be placed in an occupied box equals

(a) $\frac{4}{5}\left(\frac{3}{5}\right)^2$ (b) $\left(\frac{3}{5}\right)^3$ (c) $\left(\frac{3}{5}\right)^2$ (d) $\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$

Ans. (c)

PART-C

41. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map satisfying $\|f(x)-f(y)\| \geq \|x-y\|$, for all $x, y \in \mathbb{R}^n$. Then

(a) f is onto

(b) $f(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^n$

(c) $f(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$

(d) $f(0) = 0$

Ans. (a)

42. Let $f:\mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x) = x^t Ax$, where A is a 4×4 matrix with real entries and x^t denotes the transpose of x. The gradient of at a point x_0 necessarily is

(a) $2Ax_0$ (b) $Ax_0 + A^t x_0$ (c) $2A^t x_0$ (d) Ax_0

Ans. (a,c,d)

43. Let $f(x,y) = \frac{1-\cos(x+y)}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$f(0,0) = \frac{1}{2}$

and $g(x,y) = \frac{1-\cos(x+y)}{(x+y)^2}$ if $x+y \neq 0$

$g(x,y) = \frac{1}{2}$ if $x+y=0$

Then

(a) f is continuous at (0, 0)

(b) f is continuous everywhere except at (0, 0)

(c) g is continuous at (0, 0)

(d) g is continuous everywhere

Ans. (c,d)

44. Evaluate $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n}{k^2+n^2}$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{8}$

Ans. (c,d)

45. Consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the usual metric. Suppose a and b are irrational numbers with $a < b$ and let $K = [a,b] \cap \mathbb{Q}$. Then

(a) K is a bounded subset of $\mathbb{Q}$

(b) K is a closed subset of $\mathbb{Q}$

(c) K is a compact subset of $\mathbb{Q}$

(d) K is an open subset of $\mathbb{Q}$

Ans. (a,b,d)

46. Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying $f(x+y)=f(x)f(y), \forall x,y\in\mathbb{R}$ and $\lim_{x\to 0} f(x)=1$. Which of the following are necessarily true ?

(a) f is stricly increasing

(b) f is either constant or bounded

(c) $f(rx)=f(x)^r$ for every rational $r\in\mathbb{Q}$

(d) $f(x)\geq 0, \forall x\in\mathbb{R}$

Ans. (d)

47. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Q}$ the set of all rational numbers. For $0\leq\in\leq\frac{1}{2}$, let $A_\in$ be the open interval $(0,1-\in)$. Which of the following are true ?

(a) $\sup_{0<\in<\frac{1}{2}} \sup(A_\in)<1$

(b) $0<\in_1<\in_2<\frac{1}{2}\Rightarrow \inf(A_{\in_1})<\inf(A_{\in_2})$

(c) $0<\in_1<\in_2<\frac{1}{2}\Rightarrow \sup(A_{\in_1})>\sup(A_{\in_2})$

(d) $\sup(A_\in\cap\mathbb{Q})=\sup(A_\in\cap(\mathbb{R}\setminus\mathbb{Q}))$

Ans. (b,c,d)

48. Let $a_{mn}, m\geq 1, n\geq 1$ be a double array of real numbers. Define

$P=\liminf_{n\to\infty}\liminf_{m\to\infty} a_{mn}$, $Q=\liminf_{n\to\infty}\limsup_{m\to\infty} a_{mn}$

$R=\limsup_{n\to\infty}\liminf_{m\to\infty} a_{mn}$, $S=\limsup_{n\to\infty}\limsup_{m\to\infty} a_{mn}$

Which of the following statements are necessarily true?

(a) $P\leq Q$ (b) $Q\leq R$ (c) $R\leq S$ (d) $P\leq S$

Ans. (b)

49. Which of the following are convergent?

(a) $\sum_{n=1}^{\infty} n^2 2^{-n}$ (b) $\sum_{n=1}^{\infty} n^{-2} 2^{n}$ (c) $\sum_{n=1}^{\infty}\frac{1}{n\log n}$ (d) $\sum_{n=1}^{\infty}\frac{1}{n\log(1+1/n)}$

Ans. (a,b,c)

50. Let A be an $m\times n$ matrix of rank m with $n>m$. If for some non-zero real number α, we have $x^tAA^tx=\alpha x^tx$, for all $x\in\mathbb{R}^m$ then A^tA has

(a) exactly two distinct eigenvalues

(b) 0 as an eigenvalue with multiplicity $n-m$

(c) α as a non-zero eigenvalue

(d) exactly two non-zero distinct eigenvalues

Ans. (a,b,c)

51. For every 4×4 real symmetric non-singular matrix , there exists a positive integer p such that

(a) $pI + A$ is positive definite
(b) A^p is positive definite
(c) A^{-p} is positive definite
(d) exp (pA) – I is positive definite

Ans. (c,d)

52. Let V be the vector space over $\mathbb{C}$ of all polynomials in a variable X of degree at most 3. Let $D: V \to V$ be the linear operator given by differentiation with respect to X. Let A be the matrix of D with respect to some basis for V. Which of the following are true?

(a) A is a nilpotent matrix
(b) A is a diagonalizable matrix
(c) the rank of A is 2
(d) the Jordan canonical form of A is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ans. (a,d)

53. Let A be 3×3 matrix with real entries. Identify the correct statements.

(a) A is necessarily diagonalizable over $\mathbb{R}$
(b) if A has distinct real eigenvalues then it is diagonalizable over $\mathbb{R}$
(c) if A has distinct eigenvalues then it is diagonalizable over $\mathbb{C}$
(d) if all eigenvalues of A are non-zero then it is diagonalizable over $\mathbb{C}$

Ans. (a,d)

54. Let M = { A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Bigg| \, a,b,c,d \in \mathbb{Z}$ and the eigenvalues of A are in $\mathbb{Q}$}. Then

(a) M is empty
(b) M = {$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Bigg| \, a,b,c,d \in \mathbb{Z}$ }
(c) if $A \in M$ then the eigenvalues of A are in $\mathbb{Z}$
(d) if $A, B \in M$ are such that AB = I then det $A \in \{+1, -1\}$

Ans. (c,d)

55. Let $f : [-1,1] \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then

(a) f is of bounded variation on [–1, 1]
(b) $f\,'$ is of bounded variation on [–1, 1]
(c) $|f'(x)| \le 1 \quad \forall x \in [-1,1]$
(d) $|f'(x)| \le 3 \; \forall x \in [-1,1]$

Ans. (b,c)

56. Let A be an $m \times n$ matrix with rank r. If the linear system $AX = b$ has a solution for each $b \in \mathbb{R}^m$, then

(a) $m = r$

(b) the column space of A is a proper subspace of $\mathbb{R}^m$

(c) the null space of A is a non-trivial subspace of $\mathbb{R}^n$ whenever $m = n$

(d) $m \geq n$ implies $m = n$

Ans. **(a,d)**

57. Let $\ell^2 = \left\{x = (x_n)_{n \geq 1} \mid x_n \in \mathbb{R}, \Sigma_{n=1}^{\infty} x_n^2 < \infty\right\}$ be the Hilbert space of square summable sequences and let e_k denote the k^{th} co-ordinate vector (with 1 in k^{th} place, 0 elsewhere). Which of the following subspaces is NOT dense in ℓ^2 ?

(a) span $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, ...\}$

(b) span $\{2e_1 - e_2, 2e_2 - e_3, 2e_3 - e_4, ...\}$

(c) span $\{e_1 - 2e_2, e_2 - 2e_3, e_3 - 2e_4, ...\}$

(d) span $\{e_2, e_3, e_4, ...\}$

Ans. **(a,b,c)**

58. Consider $X = \left\{\left(x, \sin\frac{1}{x}\right) \middle| 0 < x \leq 1\right\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$ as a subspace of $\mathbb{R}^2$ and $Y = [0,1)$ as a subspace of $\mathbb{R}$. Then

(a) X is connected

(b) X is compact

(c) $X \times Y$ (in product topology) is connected

(d) $X \times Y$ (in product topology) is compact

Ans. **(a,b,c)**

59. Let X and Y be topological spaces where Y is Hausdorff. Let $X \times Y$ be given the product topology. Then for a function $f : X \to Y$ which of the following statements are necessarily true ?

(a) if f is continuous, then graph$(f) = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$

(b) if graph (f) is closed in $X \times Y$, then f is continuous

(c) if graph (f) is closed in $X \times Y$, then f need not be continuous

(d) if Y is finite, then f is continuous

Ans. **(a,c)**

60. Let d and d' be metrics on a non-empty set X. Then which of the following are metrics on X?

(a) $\rho_1(x, y) = d(x, y) + d'(x, y)$ for all $x, y \in X$

(b) $\rho_2(x, y) = d(x, y) d'(x, y)$ for all $x, y \in X$

(c) $\rho_3(x, y) = \max\{d(x, y) d'(x, y)\}$ for all $x, y \in X$

(d) $\rho_4(x, y) = \min\{d(x, y) d'(x, y)\}$ for all $x, y \in X$

Ans. **(a,b,c,d)**

61. Let F be a finite field and let K/F be a field extension of degree 6. Then the Galois group of K/F is isomorphic to

(a) the cyclic group of order 6

(b) the permutation group on {1, 2, 3}

(c) the permutation group on {1, 2, 3, 4, 5, 6}

(d) the permutation group on {1}

Ans. **(a,b,c)**

62. Let $z = e^{2\pi i/7}$ and $\theta = z + z^2 + z^4$ then

(a) $\theta \in \mathbb{Q}$

(b) $\theta \in \mathbb{Q}\left(\sqrt{D}\right)$ for some D > 0

(c) $\theta \in \mathbb{Q}\left(\sqrt{D}\right)$ for some D < 0

(d) $\theta \in i\mathbb{R}$

Ans. **(a,c)**

63. For any prime number p, let A_p be the set of integers $d \in \{1, 2, \ldots, 999\}$ such that the power of p in the prime factorisation of d is odd. Then the cardinality of

(a) A_3 is 250 (b) A_5 is 160 (c) A_7 is 124 (d) A_{11} is 82

Ans. **(a,b,c,d)**

64. Which of the following rings are principal ideal domains (PIDs) ?

(a) $\mathbb{Z}[X]/\langle X^2+1\rangle$ (b) $\mathbb{Z}[X]$

(c) $\mathbb{C}[X,Y]$ (d) $\mathbb{R}[X,Y]/\langle X^2+1, Y\rangle$

Ans. **(c)**

65. Let G be a finite abelian group and $a, b \in G$ with order $(a) = m$, order $(b) = n$. Which of the following are necessarily true ?

(a) order $(ab) = mn$

(b) order (ab) = $lcm(m,n)$

(c) there is an element of G whose order is 1cm (m,n)

(d) order $(ab) = gcd(m,n)$

Ans. **(a,d)**

66. For a set X, let $P(X)$ be the set of all subsets of X and let $\Omega(X)$ be the set of all functions $f : X \to \{0,1\}$. Then

(a) if X is finite then $P(X)$ is finite

(b) if X and Y are finite sets and if there is a 1–1 correspondence between $P(X)$ and $P(Y)$, then there is a 1–1 correspondence between X and Y

(c) there is no 1–1 correspondence between X and $P(X)$

(d) there is a 1–1 correspondence between Ω(X) and $P(X)$

Ans. **(a,c,d)**

67. Let f be a non-constant entire function and let E be the image of f. Then

(a) E is an open set (b) $E \cap \{z : |z| < 1\}$ is empty

(c) $E \cap \mathbb{R}$ is non-empty (d) E is a bounded set

Ans. **(c)**

68. LetLet $p(z)=z^n+a_{n-1}z^{n-1}+....+a_0$, where $a_0,....,a_{n-1}$ are complex numbers and let

$q(z)=1+a_{n-1}z+...+a_0z^n$. If $|p(z)|\le 1$ for all z with $|z|\le 1$ then

(a) $|q(z)|\le 1$ for all z with $|z|\le 1$

(b) $q(z)$ is a constant polynomial

(c) $p(z)=z^n$ for all complex numbers

(d) $p(z)$ is a constant polynomial

Ans. **(a)**

69. Let $f:\mathbb{C}\to\mathbb{C}$ be a holomorphic function and let u be the real part of f and v the imaginary part of f. Then, for $x,y\in\mathbb{R}, |f'(x+iy)|^2$, is equal to

(a) $u_x^2+u_y^2$ (b) $u_x^2+v_x^2$ (c) $v_y^2+u_y^2$ (d) $v_y^2+v_x^2$

Ans. **(a,c)**

70. Let f be an entire function. Consider $A=\{z\in\mathbb{C}\mid f^{(n)}(z)=0\}$ for some positive integer . Then

(a) if $A=\mathbb{C}$, then f is a polynomial

(b) if $A=\mathbb{C}$, then f is a constant function

(c) if A is uncountable, then f is a polynomial

(d) if A is uncountable, then f is a constant function

Ans. **(a,c)**

71. Let $B=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2<1\}$, and let

$C_{Id}^2(\bar{B};\mathbb{R}^2)=\{u\in C^2(\bar{B};\mathbb{R}^2)\mid u(x_1,x_2)=(x_1,x_2), \text{for } (x_1,x_2)\in\partial B\}$. Let $u=(u_1,u_2)$ and define

$J:C_{Id}^2(\bar{B};\mathbb{R}^2)\to\mathbb{R}$ by $J(u)=\int_B\left(\frac{\partial u_1\partial u_2}{\partial x_1\partial x_2}-\frac{\partial u_1\partial u_2}{\partial x_1\partial x_2}\right)dx_1dx_2$. Then

(a) $\inf\{J(u):u\in C_{Id}^2(\bar{B}:\mathbb{R}^2)\}=0$

(b) $J(u)>0,$ for all $u\in C_{Id}^2(\bar{B}:\mathbb{R}^2)$

(c) $J(u)=1,$ for infinitely many $u\in C_{Id}^2(\bar{B};\mathbb{R})$

(d) $J(u)=\pi,$ for all $u\in C_{Id}^2(\bar{B};\mathbb{R}^2)$

Ans. **(c,d)**

72. Let φ be the solution of the integral equation $\frac{1}{2}\varphi(x)-\int_0^1 e^{x-y}\varphi(y)\,dy=x^2 \quad 0\le x\le 1$

then

(a) $\varphi(0)=20e^{-1}-8$

(b) $\varphi(0)=20e-8$

(c) $\varphi(1)=22-8e$

(d) $\varphi(1)=22-8e^{-1}$

Ans. **(a,b,c,d)**

73. Consider a non-zero, real-valued polynomial function $p(x) = a_0 + a_1 x + a_2 x^2$ of degree at most 2. Let $y = y(x)$ be a solution of the integral equation

$$y = p(x) + \int_0^x y(t)\sin(x-t)\,dt$$

Which of the following statements are necessarily correct?

(a) $y(x)$ is a polynomial function of degree ≤ 2

(b) $y(x)$ is a polynomial function of degree ≤ 4

(c) If $a_1 \ne 0$ and $a_0 + 2a_2 = 0$ then $y'(0) = 0$

(d) If $a_1 \ne 0$ and $a_0 + 2a_2 = 0$, then $y''(0) = 0$

Ans. **(a,c)**

74. Let $I : C^1[0,1] \to \mathbb{R}$ be defined as

$$I(u) := \frac{1}{2}\int_0^1 \left(u'(t)^2 - 4\pi^2 u(t)^2\right)dt$$

Let us set $(P)\, m := \inf\left\{I(u) : u \in C^1[0,1] : u(0) = u(1) = 0\right\}$

Let $\bar{u} \in C^1[0,1]$ satisfy the Euler-Lagrange Equation associated with (P). Then

(a) $m = -\infty$ i.e. I is not bounded below

(b) $m \in \mathbb{R}$, with $I(\bar{u}) = m$

(c) $m \in \mathbb{R}$, with $I(\bar{u}) > m$

(d) $m \in \mathbb{R}$, with $I(\bar{u}) < m$

Ans. **(a,b,d)**

75. Let $X = \{u \in C^1[0,1] \mid u(0) = 0\}$ and let $I : X \to \mathbb{R}$ be defined as

$$I(u) = \int_0^1 \left(u'(t)^2\right)dt$$

Which of the following are correct?

(a) I is bounded below

(b) I is not bounded below

(c) I attains its infimum

(d) I does not attain its infimum

Ans. **(b,c)**

76. For $f \in C[0,1]$ and $n > 1$, let $T(f) = \frac{1}{n}\left[\frac{1}{2}f(0) + \frac{1}{2}f(1) + \Sigma_{j=1}^{n-1} f\left(\frac{j}{n}\right)\right]$ be an approximation of the integral

$I(f) = \int_0^1 f(x)\,dx$. For which of the following functions f is $T(f) = I(f)$?

(a) $1 + \sin 2\pi nx$

(b) $1 + \cos 2\pi nx$

(c) $\sin^2 2\pi nx$

(d) $\cos^2 2\pi(n+1)x$

Ans. **(b,d)**

77. Consider the linear system $Ax = b$ with

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -2 \\ -3 & -2 & 1 \end{bmatrix}$$

Let x_n denote the n^{th} Gauss-Seidel iteration and $e_n = x_n - x$. Let M be the corresponding matrix such that $e_{n+1} = Me_n, n \geq 0$. Which of the following statements are necessarily true?

(a) all eigenvalues of M have absolute value less than 1

(b) there is an eigenvalue of M with absolute value at least 1

(c) e_n converges to 0 as $n \to \infty$ for all $b \in \mathbb{R}^3$ and any e_0

(d) e_n does not converge to 0 as $n \to \infty$ for any $b \in \mathbb{R}^3$ unless $e_0 = 0$

Ans. (a)

78. Consider the second order PDE

$$8\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0$$

Then which of the following are correct ?

(a) the equation is elliptic

(b) the equation is hyperbolic

(c) the general solution is $z = f\left(y - \frac{x}{2}\right) + g\left(y + \frac{3x}{4}\right)$, for arbitrary differentiable functions f and g

(d) the general solution is $z = f\left(y + \frac{x}{2}\right) + g\left(y - \frac{3x}{4}\right)$, for arbitrary differentiable functions f and g

Ans. (a, c)

79. Consider the Lagrange equation $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$. Then the general solution of the given equation is

(a) $F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$ for an arbitrary differentiable function F

(b) $F\left(\frac{x-y}{z}, \frac{1}{x} - \frac{1}{y}\right) = 0$ for an arbitrary differentiable function F

(c) $z = f\left(\frac{1}{x} - \frac{1}{y}\right)$ for an arbitrary differentiable function f

(d) $z = xy\ f\left(\frac{1}{x} - \frac{1}{y}\right)$ for an arbitrary differentiable function f

Ans. (a)

80. Consider a boundary value problem (BVP) $\dfrac{d^2y}{dx^2} = f(x)$ with boundary conditions $y(0) = y(1) = y'(1)$, where f is a real-valued continuous function on [0, 1]. Then which of the following are true?

(a) the given BVP has a unique solution for every f

(b) the given BVP does not have a unique solution for some f

(c) $y(x) = \int_0^x xt\ f(t)\,dt + \int_x^1 (t - x + xt)\ f(t)\,dt$ is a solution of the given BVP

(d) $y(x) = \int_0^x (x - t + xt)\ f(t)\,dt + \int_x^1 xtf(t)\,dt$ is a solution of the given BVP

Ans. **(b,d)**

81. Consider the differential equation $\dfrac{d^2y}{dx^2} - 2\tan x\dfrac{dy}{dx} - y = 0$ defined on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Which among the following are true?

(a) there is exactly one solution $y = y(x)$ with $y(0) = y'(0) = 1$ and $y\left(\dfrac{\pi}{3}\right) = 2\left(1 + \dfrac{\pi}{3}\right)$

(b) there is exactly one solution $y = y(x)$ with $y(0) = 1, y'(0) = -1$ and $y\left(-\dfrac{\pi}{3}\right) = 2\left(1 + \dfrac{\pi}{3}\right)$

(c) any solution $y = y(x)$ satisfies $y''(0) = y(0)$

(d) if y_1 and y_2 are any two solutions then $(ax + b)y_1 = (cx + d)y_2$ for some $a, b, c, d \in \mathbb{R}$

Ans. **(a,c)**

82. Consider a system of first order differential equations

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t) + y(t) \\ -y(t) \end{bmatrix}$$

The solution space is spanned by

(a) $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$ and $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \cos ht \\ e^{-t} \end{bmatrix}$

(c) $\begin{bmatrix} e^{-t} \\ -2\,e^{-1} \end{bmatrix}$ and $\begin{bmatrix} \sin ht \\ e^{-t} \end{bmatrix}$

(d) $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} e^t - \frac{1}{2}e^{-1} \\ e^{-t} \end{bmatrix}$

Ans. **(b,d)**