CSIR-NET-(P YQ) MATHEMATICAL SCIENCE
DECEMBER-2023-II

PART-B

21. Consider the following subset of $\mathbb{R}$:

Which one of the following statements is true?

(a) inf U = 5 (b) inf U = 4 (c) inf U = 3 (d) inf U = 2

Ans. (c)

22. Let X be a non-empty finite set and $Y = \{f^{-1}(0) : f$ is a real-valued function on $X\}$

Which one of the following statements is true?

(a) Y is an infinite set

(b) Y has $2^{|x|}$ elements

(c) There is a bijective function from X to Y

(d) There is a surjective function from X to Y

Ans. (b)

23. Consider the following infinite series:

(a) $\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{\sqrt{n}}$, (b) $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n^2}\right)$

Which one of the following statements is true?

(a) (a) is convergent, but (b) is not convergent (b) (a) is not convergent, but (b) is convergent.

(c) Both (a) and (b) are convergent (d) Neither (a) nor (b) is convergent.

Ans. (c)

24. Consider the sequence $(a_n)_{n\geq 1}$, where $a_n = \cos\left((-1)^n \frac{n\pi}{2} + \frac{n\pi}{3}\right)$. Which one of the following statements is true ?

(a) $\limsup_{n\to\infty} a_n = \frac{\sqrt{3}}{2}$ (b) $\limsup_{n\to\infty} a_{2n} = 1$ (c) $\limsup_{n\to\infty} a_{2n} = \frac{1}{2}$ (d) $\limsup_{n\to\infty} a_{3n} = 0$

Ans. (b)

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f and its derivative f' have no common zeros in [0,1]. Which one of the following statements is true?

(a) f never vanishes in [0,1]

(b) f has at most finitely many zeros in [0,1]

(c) f has infinitely many zeros in [0,1]

(d) $f(1/2) = 0$

Ans. (b)

26. Let $f(x)$ be a cubic polynomial with real coefficients. Suppose that $f(x)$ has exactly one real root and that this root is simple. Which one of the following statements holds for ALL antiderivatives $F(x)$ of $f(x)$?

(a) $F(x)$ has exactly one real root

(b) $F(x)$ has exactly four real roots

(c) $F(x)$ has at most two real roots

(d) $F(x)$ has at most one real root

Ans. (c)

27. We denote by I_n the $n \times n$ identity matrix. Which one of the follawing statements is true?

(a) If A is a real 3×2 matrix and B is areal 2×3 matrix such that BA = I_2 then $AB = I_3$,.

(b) Let A be the real matrix $\begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$ Then there is a matrix B with integer entries such that $AB = I_2$

(c) Let A be the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ with entries in $\mathbb{Z}/6\mathbb{Z}$. Then there is a matrix B with entries in $\mathbb{Z}/6\mathbb{Z}$ such that $AB = I_2$

(d) If A is a real non-zero 3×3 diagonal matrix, then there is a real matrix B such that $AB = I_3$

Ans. (c)

28. Let $A = (a_{i,j})$ be the n x n real matrix with $a_{i,j} = ij$ for all $1 \le i, j, j \le n$. If $n \ge 3$. which one of the following is an eigenvalue of A?

(a) 1

(b) n

(c) $n(n+1)/2$

(d) $n(n+1)(2n+1)/6$

Ans. (d)

29. Which one of the following statements is FALSE?

(a) The product of two 2×2 real matrices of rank 2 is of rank 2

(b) The product of two 3×3 real matrices of rank 2 is of rank at most 2

(c) The product of two 3×3 real matrices of rank 2 is of rank at least 2

(d) The product of two 2×2 real matrices of rank 1 can be the zero matrix

Ans. (c)

30. Let A be an $n \times n$ matrix with complex entries. If $n \ge 4$, which one of the following statement is True ?

(a) A does not have any non-zero invariant subspace in $\mathbb{C}^n$.

(b) A has an invariant subspace in $\mathbb{C}^n$ of dimension $n-3$.

(c) All eigenvalues of A are real numbers.

(d) A^2 does not have any invariant subspace in $\mathbb{C}^n$ of dimension $n-1$.

Ans. (b)

31. Let (–,–) be a symmetric bilinear form on $\mathbb{R}^2$ such that there exist nonzero $v, w \in \mathbb{R}^2$?

such that $(v,v) > 0 > (w,w)$ and $(v,w) = 0$. Let A be the 2×2 real symmetric matrix representing this bilinear form with respect to the standard basis. Which one of the following statements is true ?

(a) $A^2 = 0$

(b) rank A= 1

(c) rank A = 0

(d) There exists $u \in \mathbb{R}^2$, $u \neq 0$ such that $(u,u) = 0$

Ans. (d)

32. For $a \in \mathbb{R}$, let $A_a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & a \end{pmatrix}$. Which one of the following statements is true?

(a) A_a is positive definite for all $a < 3$

(b) A_a is positive definite for all $a > 3$

(c) A_a is positive definite for all $a \geq -2$

(d) A_a is positive definite only for finitely many values of a.

Ans. **(b)**

33. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ denote the upper half plane and let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = e^{iz}$.

Which one of the following statements is true?

(a) $f(\mathbb{H}) = \mathbb{C} \setminus \{0\}$

(b) $f(\mathbb{H}) \cap \mathbb{H}$ is countable.

(c) $f(\mathbb{H})$ is bounded

(d) $f(\mathbb{H})$ is a convex subset of $\mathbb{C}$.

Ans. **(c)**

34. Let f be a meromorphic function on an open set containing the unit circle C and its interior. Suppose that f has no zeros and no poles on C, and let n_p and n_0 denote the number of poles and zeros of f inside C, respectively. Which one of the following is true ?

(a) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}\,dz = n_0 - n_p + 1$

(b) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}\,dz = n_0 - n_p - 1$

(c) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}\,dz = n_0 - n_p$

(d) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}\,dz = n_p - n_0$

Ans. **(a)**

35. Let f: $\mathbb{C} \to \mathbb{C}$ be a real-differentiable function. Define u,v: $\mathbb{R}^2 \to \mathbb{R}$ by

$u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$, $x, y \in \mathbb{R}$

Let $\nabla u = (u_x, y_y)$ denote the gradient. Which one of the following is necessarily true ?

(a) For $c_1, c_2 \in \mathbb{C}$, the level curves $u = c_1$, and $v = c_2$, are orthogonal wherever they intersect.

(b) $\nabla u \cdot \nabla v = 0$ at every point.

(c) If f is an entire function, then $\nabla u \cdot \nabla v = 0$ at every point.

(d) If $\nabla u \cdot \nabla v = 0$ at every point, then f is an entire function.

Ans. **(c)**

36. How many roots does the polynomiaol

$z^{100}-50z^{30}+40z^{10}+6z+1$

Here in the open disc $\{z\in\mathbb{C}:|z|<1\}$?

(a) 100 (b) 50 (c) 30 (d) 0

Ans. (c)

37. In any class of 50 students, which one of the following statements is necessarily true?

(a) Two students have the same birthday

(b) Every month has birthdays of at least five students

(c) There exists a month which has birthdays of at least five students

(d) The birthdays of at least 25 students are during the first six months (from January till June)

Ans. (c)

38. Let G be any finite group. Which one of the following is necessarily true'?

(a) G is a union of proper subgroups

(b) G is a union of proper subgroups if $|G|$ has at leat two distinct prime divisors

(c) lf G is abelian, then G is a union of proper subgroups

(d) G is a union of proper subgroups if and only if G is not cyclic

Ans. (d)

39. Which one of the following is equal to $1^{37}+2^{37}+3^{37}+.......+88^{37}$ in $\mathbb{Z}/89\mathbb{Z}$?

(a) 88 (b) –88 (c) –2 (d) 0

Ans. (d)

40. Consider the field C together with the Euclidean topology. Let K be a proper subfield of C that is not contained in $\mathbb{R}$. Which one of the following statements is hecessarily true ?

(a) K is dense in $\mathbb{C}$.

(b) K is an algebraic extension of $\mathbb{Q}$.

(c) $\mathbb{C}$ is an algebraic extension of K.

(d) The smallest closed subset of $\mathbb{C}$ containing K is NOT a field

Ans. (a)

41. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}(1-x)^2\sin(x^2), & x\in(0,1)\\ 0, & \text{otherwise}\end{cases}$$

and f' be its derivative. Let $S=\{c\in\mathbb{R}: f'(x)\le cf(x) \text{ for all } x\in\mathbb{R}\}$

Which one of the following is true ?

(a) $S=\phi$

(b) $S\neq\phi$ and Sis a proper subset of $(1,\infty)$

(c) $(2,\infty)$ is a proper subset of S

(d) $S\cap(0,1)\neq\phi$

Ans. (a)

42. The smallest real number A for which the problem

$-y''+3y=\lambda y,\ y(0)=0,\ y(\pi)=0$

has a non-trivial solution is

(a) 3 (b) 2 (c) 1 (d) 4

Ans. (d)

43. The following partial differential equation $x^2\dfrac{\partial^2 u}{\partial x^2}-2xy\dfrac{\partial^2}{\partial x\partial y}-3y^2\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial u}{\partial x}-\dfrac{\partial u}{\partial y}=0$ is

(a) elliptic in $\{(x,y)\in\mathbb{R}^2 : y>0\}$
(b) parabolic in $\{(x,y)\in\mathbb{R}^2 : x>0, y>0\}$
(c) hyperbolic in $\{(x,y)\in\mathbb{R}^2 : xy\neq 0\}$
(d) parabolic in $\{(x,y)\in\mathbb{R}^2 : xy\neq 0\}$

Ans. (c)

44. Consider the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2}-4\frac{\partial^2 u}{\partial x^2}=0, -\infty<x<\infty, \quad t>0,$$

$$u(x,0)=\begin{cases} e^{\left(-\frac{1}{x^2}\right)}, & x\neq 0 \\ 0 & x=0 \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0)=xe^{-x^2}, x\in\mathbb{R}$$

Which one of the following is true ?

(a) $\lim\limits_{t\to\infty}u(5,t)=1$
(b) $\lim\limits_{t\to\infty}u(5,t)=2$
(c) $\lim\limits_{t\to\infty}u(5,t)=\dfrac{1}{2}$
(d) $\lim\limits_{t\to\infty}u(5,t)=0$

Ans. (a)

45. Using Euler's method with the step size 0.05, the approximate value of the solution“for the initial value problem

$$\frac{dy}{dx}=\sqrt{3x+2y+1}, \quad y(1)=1$$

at $x = 1.1$ (rounded off to two decimal places), is

(a) 1.50
(b) 1.65
(c) 1.25
(d) 1.15

Ans. (c)

46. The cardinality of the set of extremals of $J[y]=\int_0^1 (y')^2\,dx$, subject to $y(0)=1, \quad y(1)=6, \quad \int_0^1 y\,dx=3$ is

(a) 0
(b) 1
(c) 2
(d) countably infinite

Ans. (b)

47. The value of λ for which the integral equation $y(x)=\lambda\int_0^1 x^2e^{x+t}y(t)\,dt$ has a non-zero solution, is

(a) $\dfrac{4}{1+e^2}$
(b) $\dfrac{2}{1+e^2}$
(c) $\dfrac{4}{e^2-1}$
(d) $\dfrac{2}{e^2-1}$

Ans. (c)

48. Let g denote the acceleration due to gravity and $a > 0$. A particle of mass m glides (without friction) on the cycloid given by $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$, with $0\le\theta\le 2\pi$. Then the equation of motion of the particle is

(a) $(1-\cos\theta)\ddot{\theta}+\dfrac{1}{2}(\sin\theta)(\dot{\theta})^2-\dfrac{g}{2a}\sin\theta=0$
(b) $(1-2\cos\theta)\ddot{\theta}+(\sin\theta)(\dot{\theta})^2-\dfrac{g}{2}\sin\theta=0$
(c) $m(1-2\cos\theta)\ddot{\theta}+(\sin\theta)(\dot{\theta})^2+\dfrac{g}{2}\sin\theta=0$
(d) $m(1-2\cos\theta)\ddot{\theta}+\dfrac{m}{2}(\sin\theta)(\dot{\theta})^2-\dfrac{g}{2}\sin\theta=0$

Ans. (a)

49. Let (X,Y) be a random vector with the joint moment generating function

$$M_{X,Y}(t_1,t_2)=\left(\frac{3}{4}+\frac{1}{4}e^{t_1}\right)^2\left(\frac{5}{6}+\frac{1}{6}e^{t_2}\right)^3, \quad (t_2,t_2)\in\mathbb{R}^2$$

Then $P(X+2Y>1)$ is equal to

(a) $\frac{1581}{3456}$ (b) $\frac{1875}{3456}$ (c) $\frac{125}{3456}$ (d) $\frac{3331}{3456}$

Ans. **(a)**

60. Consider the linear programming prablem:

Maximize $z = 3x + 4y$

subjectto $x+y \le 12$, $2x+3y \le 30$, $x + 4y \le 36$, $x \ge 0, y \ge 0$.

Then the optimal solution of the given problem is

(a) $x^* = 6$, $y^* = 6$ (b) $x^* = 7$, $y^* = 5$ (c) $x^* = 3$, $y^* = 8$ (d) $x^* = 4$, $y^* = 8$

Ans. **(a)**

PART-C

61. Let $\{A_n\}_{n\ge1}$ be a collection of non-empty subsets of $\mathbb{Z}$ such that $A_n \cap A_m = \phi$ for $m \ne n$. If $\mathbb{Z} = \bigcup_{n\ge1} A_n$, then which of the following statements are necessarily true ?

(a) A_n is finite for every integer $n \ge 1$.

(b) A_n is finite for same integer $n \ge 1$.

(c) A_n is infinite for some integer $n \ge 1$.

(d) A_n is countable (finite or infinite) for every integer $n \ge 1$

Ans. **(d)**

62. Let x be a real number. Which of the following statements are true ?

(a) There exists an integer $n \ge 1$ such that $n^2 \sin\frac{1}{n} \ge x$

(b) There exists an integer $n \ge 1$ such that $n\cos\frac{1}{n} \ge x$

(c) There exists an integer $n \ge 1$ such that $ne^{-n} \ge x$

(d) There exists an integer $n \ge 2$ such that $n(\log n)^{-1} \ge x$

Ans. **(a, b, d)**

63. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $|f(x)-f(y)| \ge \log(1+|x-y|)$ for all $x, y \in \mathbb{R}$. Which of the following statements are true ?

(a) f is necessarily one-one (b) f need not be one-one

(c) f is necessarily onto (d) f need not be onto

Ans. **(a,c)**

64. Let $f:[0,\infty]\to\mathbb{R}$ be the periodic function of period 1 given by $f(x)=1-|2x-1|$ for $x\in[0,1]$

Further, define $g:[0,\infty)\to\mathbb{R}$ by $g(x)=f(x^2)$. Which of the following statements are true ?

(a) f is continuous on $[0, \infty)$ (b) f is uniformly continuous on $[0, \infty)$

(c) g is continuous on $[0, \infty)$ (d) g is uniformly continuous on $[0, \infty)$

Ans. **(a,b,c)**

65. Let $(f_n)_{n\geq 1}$ be the sequence of functions defined on [0,1] by

$$f_n(x) = x^n \log\left(\frac{1+\sqrt{x}}{2}\right)$$

Which of the following statements are true?

(a) (f_n) converges pointwise on [0,1]

(b) (f_n) converges uniformly on compact subsets of [0,1) but not on [0,1)

(c) (f_n) converges uniformly on [0,1) but not on [0,1]

(d) (f_n) converges uniformly on [0,1]

Ans. **(a,d)**

66. For a real number λ, consider the improper integrals

$$I_\lambda = \int_0^1 \frac{dx}{(1-x)^\lambda},\ K_\lambda = \int_1^\infty \frac{dx}{x^\lambda}$$

Which of the following statements are true ?

(a) There exists λ such that I_λ converges, but K_λ does not converge

(b) There exists λ such that K_λ converges, but I_λ does not converge

(c) There exists λ such that I_λ K_λ both converge

(d) There exists λ such that neither I_λ nor K_λ converges

Ans. **(a, b, d)**

67. Which of the following statements are true ?

(a) The function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} [x]\sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

has a discontinuity at 0 which is removable

(b) The function $f:[0,\infty)\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(\log x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

has a discontinuity at 0 which is NOT removable.

(c) The function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{1/x} & \text{for } x < 0 \\ e^{1/(x+1)} & \text{for } x \geq 0 \end{cases}$$

has a jump discontinuity at 0

(d) Let $f,g:[0,1]\to\mathbb{R}$ be two functions of bounded variation. Then the product fg has at most countably many discontinuities

Ans. **(b,c,d)**

68. For real numbers a,b,c,d,e,f, consider the function $F:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$F(x,y) = (ax+by+c,\ dx+ey+f),\ \text{ for } x,y\in\mathbb{R}$$

Which of the following statements are true?

(a) F is continuous.

(b) F is uniformly continuous.

(c) F is differentiable.

(d) F has partial derivatives of all orders

Ans. **(a,b,c,d)**

69. For a differentiable surjective function $f:(0,1)\to(0,1)$, consider the function

$F:(0,1)\times(0,1)\to(0,1)\times(0,1)$ given by

$F(x,y)=(f(x),f(y)),\quad x,y\in(0,1)$. If $f'(x)\neq 0$ for every $x\in(0,1)$, then which of the following statements are true ?

(a) F is injective

(b) F is increasing

(c) For every $(x',y')\in(0,1)\times(0,1)$, there exists a unique $(x,y)\in(0,1)\times(0,1)$ such that $F(x,y)=(x',y')$.

(d) The total derivative $DF(x,y)$ is invertible for all $(x,y)\in(0,1)\times(0,1)$

Ans. **(a),(c),(d)**

70. Suppose that $f:[-1,1]\to\mathbb{R}$ is continuous. Which of the following imply that f is identically zero on $[-1,1]$?

(a) $\int_{-1}^{1} f(x)x^n dx=0$ for all $n\geq 0$

(b) $\int_{-1}^{1} f(x)p(x)dx=0$ for all real polynomials $p(x)$

(c) $\int_{-1}^{1} f(x)x^n dx=0$ forall $n\geq 0$ odd.

(d) $\int_{-1}^{1} f(x)x^n dx=0$ for all $n\geq 0$ even.

Ans. **(a,b)**

71. Let $\mathbb{F}$ be a finite field and *V'* be a finite dimensional non-zero $\mathbb{F}$-vector space. Which of the following can NEVER be true ?

(a) V is the union of 2 proper subspaces.

(b) V is the union of 3 proper subspaces.

(c) V has a unique basis.

(d) V has precisely two bases.

Ans. **(a)**

72. Se Let $T:\mathbb{R}^5\to\mathbb{R}^5$ be a $\mathbb{R}$-linear transformation. Suppose that (1, –1, 2, 4, 0), (4, 6, 1, 6, 0) and (5, 5, 3, 9, 0) span the null space of T. Which of the following statements are true ?

(a) The rank of T is equal to 2

(b) Suppose that for every vector $v\in\mathbb{R}^5$, there exists n such that $T_v^n=0$. Then T^2 must be zero

(c) Suppose that for every vector $v\in\mathbb{R}^5$, there exists n such that $T_v^n=0$. Then T^3 must be zero

(d) (–2,–8, 3, 2, 0) is contained in the null space of T

Ans. **(a,c,d)**

73. Let X,Y be two $n\times n$ real matrices such that

$XY=X^2+X+I$

Which of the following statements are necessarily true ?

(a) X is invertible (b) $X+I$ is invertible (c) $XY=YX$ (d) Y is invertible

Ans. **(a,c)**

74. Consider $A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$. Suppose $A^5-4A^2-7A^3+11A^2-A-10I=aA+bI$ for some $a,b\in\mathbb{Z}$.

Which of the following statements are true ?

(a) $a+b>8$ (b) $a+b<7$ (c) $a+b$ is divisible by 2 (d) $a>b$

Ans. **(b,c)**

75. Let A be an $n\times n$ real symmetric matrix. Which of the following statements are necessarily true ?

(a) A is diagonalizable.

(b) lf $A^k = I$ for some positive integer k, then $A^2 = I$.

(c) If $A^k = 0$ for some positive integer k, then $A^2 = 0$.

(d) All eigenvalues of A are real

Ans. **(a,b,c,d)**

76. Suppose a 7×7 block diagonal complex A has blacks

$$(0),(1),\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \text{ and } \begin{pmatrix}2\pi i & 1 & 0\\ 0 & 2\pi i & 0\\ 0 & 0 & 2\pi i\end{pmatrix} \text{ along the diagonal}$$

Which of the following statements are true ?

(a) The characteristic polynomial of A is $x^3(x-1)(x-2\pi i)^3$

(b) The minimal polynomial of A is $x^2(x-1)(x-2\pi i)^3$.

(c) The dimensions of the eigenspaces for 0, 1, $2\pi i$ are 2, 1, 3 respectively

(d) The dimensions of the eigenspaces for 0 , 1, 2πi are 2, 1, 2 respectively

Ans. **(a,d)**

77. Let A be a real diagonal matrix with characteristic polynomial $\lambda^3 - 2\lambda^2 - \lambda + 2$. Define a bilinear form $\langle v, w\rangle = v^t Aw$ on $\mathbb{R}^3$. Which of the following statements are true ?

(a) A is positive definite.

(b) A^2 is positive definite.

(c) There exists a non zero $v \in \mathbb{R}^3$ such that $\langle v, v\rangle = 0$.

(d) rank $A = 2$

Ans. **(b,c)**

78. Consider the quadratic form $Q(x, y, z) = x^2 + xy + y^2 + xz + yz + z^2$. Which of the following statements are true ?

(a) There exists a non-zero $u \in \mathbb{Q}^3$ such that $Q(u) = 0$.

(b) There exists a non-zero $u \in \mathbb{R}^3$ such that $Q(u) = 0$.

(c) There exist a non-zero $u \in \mathbb{C}^3$ such that $Q(u) = 0$.

(d) The real symmetric 3×3 matrix A which satisfies

$$Q(x,y,z) = [x\ y\ z]A\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{for all } x, y, z \in \mathbb{R} \text{ is invertible.}$$

Ans. **(c, d)**

79. Let X be an uncountable subset of $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that for every $z \in X$, there exists an integer $n \geq 1$ such that $f^{(n)}(z) = 0$. Which of the following statements are necessarily tue?

(a) $f = 0$.

(b) f is a constant function.

(c) There exists a compact subset K of $\mathbb{C}$ such that f^{-1}(k) is not compact

(d) f is a polynomial

Ans. **(d)**

80. Let $\Omega_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\Omega_2 = \mathbb{C}$. Which of the following statements are true ?

(a) There exists a holomorphic surjective map $f : \Omega_1 \Rightarrow \Omega_2$

(b) There exists a holomorphic surjective map $f : \Omega_2 \Rightarrow \Omega_1$

(c) There exists a holomorphic injective map $f : \Omega_1 \Rightarrow \Omega_2$

(d) There exists a holomorphic injective map $f : \Omega_2 \Rightarrow \Omega_1$

Ans. (a,c)

81. For an integer *k*, consider the contour integral $I_k = \int_{|z|=1} \frac{e^z}{z^k}$. Which of the following statements are true ?

(a) $I_k = 0$ for every integer *k*

(b) $I_k \neq 0$ if $k \geq 1$

(c) $|I_k| \leq |I_{k+1}|$ for every integer *k*

(d) $\lim_{k\to\infty} |I_k| = \infty$

Ans. (b)

82. For every $n \geq 1$, consider the entire function $p_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$. Which of the following statements are true ?

(a) The sequence of functions $(p_n)_{n\geq 1}$ converges to an entire function uniformly on compact subsets of $\mathbb{C}$.

(b) For all $n \geq 1$, p_n has a zero in the set $\{z \in \mathbb{C} : |z| \leq 2023\}$

(c) There exists a sequence (z_n) of complex numbers such that $\lim_{n\to\infty} |z_n| = \infty$ and $p_n(z_n) = 0$ for all $n \geq 1$.

(d) Let S_n denote the set of all the zeros of p_n. If $a_n = \min_{z \in S_n} |z|$, than $a_n \to \infty$ as $n \to \infty$

Ans. (a, c, d)

83. Which of the following statements are true?

(a) Let G_1 and G_2 be finite groups such that their orders $|G_1|$ and $|G_2|$ are coprime. Then any homomorphism from G_1 to G_2 is trivial.

(b) Let *G* bea finite group. Let $f : G \to G$ be a group homomorphism such that *f* fixes more than half of the elements of *G*. Then $f(x) = x$ for all $x \in G$.

(c) Let *G* be a finite group having exactly 3 subgroups. Then G is of order p^2 for some prime *p*.

(d) Any finite abelian group *G* has at least d(|*G*|) subgroups in *G*, where d(*m*) denotes the number of positive divisors of *m*.

Ans. (a), b, c, d)

84. Let $n \in \mathbb{Z}$ be such that *n* is congruent to 1 mod 7 and n is congruent to 4 mod 15.

Which of the following statements are true ?

(a) *n* is congruent to 1 mod 3

(b) *n* is congruent to 1 mod 35

(c) *n* is congruent to 1 mod 21

(d) *n* is congruent to 1 mod 5

Ans. (a, c)

85. Let *G* be the group (under matrix multiplication) of 2×2 invertible matrices with entries from $\mathbb{Z}/9\mathbb{Z}$. Let *a* be the order of *G*. Which of the following statements are True ?

(a) *a* is divisible by 3^4

(b) *a* is divisible by 2^4

(c) *a* is not divisible by 48

(d) *a* is divisible by 3^6

Ans. (a, b)

86. Let $R = \mathbb{Z}[X]/(X^2+1)$ and $\psi : \mathbb{Z}[X] \to R$ be the natural quotient map. Which of the following statements are true?

(a) R is isomorphic to a subring of $\mathbb{C}$.

(b) For any prime number $p \in \mathbb{Z}$, the ideal generated by $\psi(p)$ is a proper ideal of R

(c) R has infinitely many prime ideals

(d) The ideal generated by $\psi(X)$ is a prime ideal in R

Ans. (a, b, c)

87. Let $f(X) = X^2 + X + 1$ and $g(X) = X^2 + X - 2$ be polynomials in $\mathbb{Z}[X]$.

Which of the following statements are true ?

(a) For all prime numbers p, $f(X)$ mod p is irreducible in $(\mathbb{Z}/p\mathbb{Z})[X]$

(b) There exists a prime number p such that $g(X)$ mod p is irreducible in $(\mathbb{Z}/p\mathbb{Z})[X]$

(c) $g(X)$ is irreducible in $\mathbb{Q}[X]$

(d) $f(X)$ is irreducible in $\mathbb{Q}[X]$

Ans. (d)

88. Let $f(X) = X^3 - 2 \in \mathbb{Q}[X]$ and let $K \subset \mathbb{C}$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Let $\omega = e^{2\pi/3}$.

Which of the following statements are true?

(a) The Galois group of K over $\mathbb{Q}$ is the symmetric group S_3.

(b) The Galois group of K over $\mathbb{Q}(\omega)$ is the symmetric group S_3.

(c) The Galois group of K over $\mathbb{Q}$ is $\mathbb{Z}/3\mathbb{Z}$.

(d) The Galois group of K over $\mathbb{Q}(\omega)$is $\mathbb{Z}/3\mathbb{Z}$

Ans. (a, d)

89. Consider $\mathbb{R}^2$ with the Euclidean topology and consider $\mathbb{Q}^2 \subset \mathbb{R}^2$? with the subspace topology. Which of the following statements are true'?

(a) $\mathbb{Q}^2$ is connected.

(b) lf A is anon-empty connected subset of $\mathbb{Q}^2$, then A has exactly one element

(c) $\mathbb{Q}^2$ is Hausdorff

(d) $\{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}$ is compact in the subspace topology.

Ans. (b, c)

90. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $p(x, y) = x$. Which of the following statements are true ?

(a) Let $A_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Then for each $\gamma \in p(A_1)$, there exists a"positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_1)$.

(b) Let $A_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$. Then for each $\gamma \in p(A_2)$, there exists a positive real number ε such that $(\gamma - \in, \gamma + \in) \subseteq p(A_2)$

(c) Let $A_3 = \{(x, y) \mathbb{R}^2 \mid xy = 0\}$. Then for each $\gamma \in p(A_3)$, there exists a"positive real number ε such that $(\gamma - \varepsilon, y + \varepsilon) \subseteq p(A_3)$.

(d) Let $A_4 = \{(x, y \in \mathbb{R}^2 \mid xy = 1)\}$. Then for each $\gamma \in p(A_4)$, there exists *a* positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_4)$

Ans. (a, c, d)

91. Consider the problem

$$y' = (1-y^2)^{10} \cos y, \quad y(0) = 0$$

Let f be the maximal interval of existence and K be the range of the solution of the"above problem. Then which of the following statements are true?

(a) $J = \mathbb{R}$ (b) $K = (-1,1)$ (c) $J = (-1,1)$ (d) $K = [-1,1]$

Ans. **(a, b)**

92. Consider the following initial value problem

$$y' = y + \frac{1}{2}\left|\sin(y^2)\right|, \quad x > 0, \ y(0) = -1$$

Which of the following statements are true ?

(a) there exists an $\alpha \in (0,\infty)$ such that $\lim_{x \to a^{-1}} |y(x)| = \infty$ (b) $y(x)$ exists on $(0, \infty)$ and it is monotone

(c) $y(x)$ exists on $(0,\infty)$, but not bounded below (d) $y(x)$ exists on $(0, \infty)$, but not bounded above

Ans. **(b, c)**

93. Cansider the initial value problem

$$x^2 y'' - 2x^2 y' + (4x-2) y = 0, \quad y(0) = 0$$

Suppose $y = \varphi(x)$ is a polynomial solution satisfying $\varphi(1) = 1$. Which of the"following statements are true ?

(a) $\varphi(4) = 16$ (b) $\varphi(2) = 2$ (c) $\varphi(5) = 25$ (d) $\varphi(3) = 3$

Ans. **(a, c)**

94. Consider the Cauchy problem

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1, (x, y) \in \mathbb{R} \times (0,\infty)$$

$$u(x,0) = kx, \quad \in x\mathbb{R},$$

with a given real parameter k. For which of the following values of k does the above problem have a solution defined on $\mathbb{R}\times(0,\infty)$?

(a) $k = 0$ (b) $k = -2$ (c) $k = 4$ (d) $k = 1$

Ans. **(a),(c),(d)**

95. Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$,

$\partial B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be its boundary and $\overline{B} = B \cup \partial B$. For $\lambda \in (0,\infty)$, let S_λ be the set of twice continuously differentiable functions in B, that are continuous on $\overline{B}$ and satisfy

$$\left(\frac{\partial u}{\partial u}\right)^2 + \lambda\left(\frac{\partial u}{\partial y}\right)^2 = 1, \text{ in B}$$

$$u(x, y) = 0 \text{ on } \partial B$$

Then which of the following statements are true?

(a) $S_1 = \phi$

(b) $S_2 = \phi$

(c) S_1 has exactly one element and S_2 has exactly two elements.

(d) S_1 and S_2 are both infinite

Ans. **(a, b)**

96. The coefficient of x^3 in the interpolating polynomial for the data

x	0	1	2	3	4
y	1	2	1	3	5

is

(a) $-\dfrac{1}{3}$ (b) $-\dfrac{1}{2}$ (c) $\dfrac{5}{6}$ (d) $\dfrac{17}{6}$

Ans. (d)

97. Consider the initial value problam

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0,$$

where f is a twice continuously differentiable function on a rectangle containing the point (x_0, y_0) With the step-size h, let the first iterate of a second order scheme to“approximate the solution of the above initial value problem be given by

$$y_1 = y_0 + Pk_1 + Qk_2,$$

where $k_1 = h\ f(x_0, y_0), k_2 = hf(x_0 + \alpha_0 h, y_0 + \beta_0 k_1)$ and $P, Q, \alpha_0, \beta_0 \in \mathbb{R}$.

Which of the following statements are correct?

(a) If $\alpha_0 = 2$, then $\beta_0 = 2,\ P = \dfrac{3}{4},\ Q = \dfrac{1}{4}$

(b) If $\beta_0 = 3$, then $\alpha_0 = 3,\ P = \dfrac{5}{6},\ Q = \dfrac{1}{6}$

(c) If $\alpha_0 = 2$, then $\beta_0 = 2,\ P = \dfrac{1}{4},\ Q = \dfrac{3}{4}$

(d) If $\beta_0 = 3$, then $\alpha_0 = 3,\ P = \dfrac{1}{6},\ Q = \dfrac{5}{6}$

Ans. (a, b)

98. Among the curves connecting the points (1,2) and (2,8), let y be the curve on which an extremal of the functional

$$J[y] = \int_1^2 (1 + x^3 y') y'\, dx$$

can be attained. Then which of the following paints lie on the curve γ ?

(a) $(\sqrt{2}, 3)$ (b) $(\sqrt{2}, 6)$ (c) $\left(\sqrt{3}, \dfrac{22}{3}\right)$ (d) $\left(\sqrt{3}, \dfrac{23}{3}\right)$

Ans. (b, c)

99. Define

$$S = \{y \in C^1[0, \pi] : y(0) = y(\pi) = 0\}$$

$$\|f\|_\infty = \max_{x \in [0,\pi]} |f(x)|, \text{ for all } f \in S$$

$$B_0(f, \varepsilon) = \{f \in S : \|f\|_\infty < \varepsilon\}$$

$$B_1(f, \varepsilon) = \{f \in S : \|f\|_\infty + \|f'\|_\infty < \varepsilon\}$$

Consider the functional $J : S \to \mathbb{R}$ given by

$$J[y] = \int_0^\pi \left(1 - (y')^2\right) y^2 dx$$

Then there exists $\varepsilon > 0$ such that

(a) $J[y] \leq J[0]$, for all $y \in B_0(0,\varepsilon)$ (b) $J[y] \leq J[0]$, for all $y \in B_1(0,\varepsilon)$

(c) $J[y] \geq J[0]$, for all $y \in B_0(0,\varepsilon)$ (d) $J[y] \geq J[0]$, for all $y \in B_1(0,\varepsilon)$

Ans. (d)

100. Consider the following Fredholm integral equation

$$y(x) - 3\int_0^1 txy(t)\,dt = f(x)$$

where $f(x)$ is a continuous function defined on the interval [0,1]. Which of the“following choices for $f(x)$ have the property that the above integral equation“admits at least one solution ?

(a) $f(x) = x^2 - \dfrac{1}{2}$ (b) $f(x) = e^x$ (c) $f(x) = 2 - 3x$ (d) $f(x) = x - 1$

Ans. (a, c)