CSIR-NET-(PYQ) MATHEMATICAL SCIENCE
DECEMBER-2016-II

PART-B

1. Consider the sets of sequences

$X = \{(x_n) : x_n \in \{0,1\}, n \in \mathbb{N}\}$

and

$Y = \{(x_n) \in X : x_n = 1$ for at most finitely many$\}$

Then

(a) X is countable, Y is finite. (b) X is uncountable, Y is countable.

(c) X is countable, Y is countable. (d) X is uncountable, Y is uncountable

Ans. (b)

2. The matrix $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ is

(a) positive definite (b) non-negative definite but not positive definite.

(c) negative definite. (d) neither negative definite nor positive definite

Ans. (a)

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$f(x, y) = (x^2, y^2 + \sin x)$

Then the derivative of f at (x, y) is the linear transformation given by

(a) $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$ (b) $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$ (c) $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

Ans. (a)

4. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = xy$. Let $v = (1, 2)$ and $a = (a_1, a_2)$ be two elements of $\mathbb{R}^2$. The directional derivative of f in the direction of v at a is

(a) $a_1 + 2a_2$ (b) $a_2 + 2a_1$ (c) $\frac{a_2}{2} + a_1$ (d) $\frac{a_1}{2} + a_2$

Ans. (b)

5. $\lim_{n\to\infty} \frac{1}{n^4} \sum_{j=0}^{2n-1} j^3$ equals

(a) 4 (b) 16 (c) 1 (d) 8

Ans. (a)

6. $f:\mathbb{R}\to\mathbb{R}$ is such that $f(0)=0$ and $\left|\frac{df}{dx}(x)\right|\le 5$ for all x. We can conclude that $f(1)$ is in

(a) $(5, 6)$ (b) $[-5,5]$ (c) $(-\infty,-5)\cup(5,\infty)$ (d) $[-4,4]$

Ans. **(b)**

7. Which of the following subsets of $\mathbb{R}^{+}$ is a basis of $\mathbb{R}^4$?

$$B_1=\{(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)\}$$

$$B_2=\{(1,0,0,0),(1,2,0,0),(1,2,3,0),(1,2,3,4)\}$$

$$B_3=\{(1,2,0,0),(0,0,1,1),(2,1,0,0),(-5,5,0,0)\}$$

(a) B_1 and B_2 but not B_3 (b) B_1, B_2 and B_3
(c) B_1 and B_3 but not B_2 (d) Only B_1

Ans. **(a)**

8. Let $D_1=\det\begin{pmatrix} a & b & c\\ x & y & z\\ p & q & r\end{pmatrix}$ and $D_2=\det\begin{pmatrix} -x & a & -p\\ y & -b & q\\ z & -c & r\end{pmatrix}$. Then

(a) $D_1=D_2$ (b) $D_1=2D_2$ (c) $D_1=-D_2$ (d) $2D_1=D_2$

Ans. **(c)**

9. Consider the matrix $A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, where $\theta=\frac{2\pi}{31}$, Then A^{2015} equals

(a) A (b) I

(c) $\begin{pmatrix}\cos 13\theta & \sin 13\theta\\ -\sin 13\theta & \cos 13\theta\end{pmatrix}$ (d) $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

Ans. **(b)**

10. Let J denote the matrix of order $n\times n$ with all entries 1 and let B be a $(3n)\times(3n)$ matrix given by

$$B=\begin{pmatrix}0 & 0 & J\\ 0 & J & 0\\ J & 0 & 0\end{pmatrix}$$

Then the rank of B is

(a) $2n$ (b) $3n-1$ (c) 2 (d) 3

Ans. **(d)**

11. Which of the following sets of functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$?

$$S_1=\left\{f \mid \lim_{x\to 3} f(x)=0\right\}$$

$$S_2=\left\{g \mid \lim_{x\to 3} g(x)=1\right\}$$

$$S_3=\left\{h \mid \lim_{x\to 3} h(x)\ \text{exists}\right\}$$

(a) Only S_1 (b) Only S_2
(c) S_1 and S_3 but not S_2 (d) All the three are vector spaces

Ans. **(c)**

12. Let A be an $n \times m$ matrix with each entry equal to +1, –1 or 0 such that every column has exactly one +1 and exactly one –1. We can conclude that

(a) Rank $A \le n-1$ (b) Rank $A = m$ (c) $n \le m$ (d) $n-1 \le m$

Ans. **(a)**

13. The radius of convergence of the series $\sum_{n=1}^{\infty} z^{n^2}$ is

(a) 0 (b) ∞ (c) 1 (d) 2

Ans. **(c)**

14. Let C be the circle $|z| = 3/2$ in the complex plane that is oriented in the counter clockwise direction. The value a for which

$$\int_C \left(\frac{z+1}{z^2 - 3z + 2} + \frac{a}{z-1} \right) dz = 0 \text{ is}$$

(a) 1 (b) –1 (c) 2 (d) –2

Ans. **(c)**

15. Suppose f and g are entire functions and $g(z) \ne 0$ for all $z \in \mathbb{C}$. If $|f(z)| \le |g(z)|$, then we conclude that

(a) $f(z) \ne 0$ for all $z \in \mathbb{C}$ (b) f is a constant function.

(c) $f(0) = 0$ (d) for some $c \in \mathbb{C},\ f(z) = Cg(z)$

Ans. **(d)**

16. Let f be a holomorphic function on $0 < |z| < \in, \in > 0$, given by a convergent Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ .Given also that $\lim_{z \to 0} |f(z)| = \infty$, We can conclude that

(a) $a_{-1} \ne 0$ and $a_{-n} = 0$ for all $n \ge 2$

(b) $a_{-N} \ne 0$ for some $N \ge 1$ and $a_{-n} = 0$ for all $n > N$

(c) $a_{-n} = 0$ for all $n \ge 1$

(d) $a_{-n} \ne 0$ for all $n \ge 1$

Ans. **(b)**

17. Given a natural number $n > 1$ such that $(n-1)! \equiv -1 (\text{mod } n)$. We can conclude that

(a) $n = p^k$ where is prime, $k > 1$.

(b) $n = pq$ where p and q are distinct primes

(c) $n = pqr$ where p, q, r are distinct primes

(d) $n = p$ where p is a prime.

Ans. **(d)**

18. Let S_n denote the permutation group on n symbols and A_n be the subgroup of even permutations. Which of the following is true ?

(a) There exists a finite group which is not a subgroup of S_n for any $n \geq 1$

(b) Every finite group is a subgroup of A_n for some $n \geq 1$

(c) Every finite group is a quotient of A_n for some $n \geq 1$

(d) No finite abelian group is a quotient of S_n for $n > 3$

Ans. (b)

19. What is the number of non-singular 3×3 matrices over $\mathbb{F}_2$, the finite field with two elements ?

(a) 168 (b) 384 (c) 2^3 (d) 3^2

Ans. (a)

20. Let G be an open set in $\mathbb{R}^n$. Two points $x, y \in G$ are said to be equivalent if they can be joined by a continuous path completely lying inside G. Number of equivalence classes is

(a) only one (b) at most finite.

(c) at most countable (d) can be finite, countable or uncountable

Ans. (c)

21. Let $\left(x(t), y(t)\right)$ satisfy the system of ODEs

$$\frac{dx}{dt} = -x + ty$$

$$\frac{dy}{dt} = tx - y$$

If $\left(x_1(t), y_1(t)\right)$ and $\left(x_2(t), y_2(t)\right)$ are two solutions and

$\Phi(t) = x_1(t)\, y_2(t) - x_2(t)\, y_1(t)$ then $\dfrac{d\Phi}{dt}$ is equal to

(a) -2Φ (b) 2Φ (c) $-\Phi$ (d) Φ

Ans. (a)

22. The boundary value problem $x^2 y'' - 2xy' + 2y = 0$, subject to the boundary conditions

$y(1) + \alpha y'(1) = 1,\ y(2) + \beta y'(2) = 2$, has a unique solution if

(a) $\alpha = -1, \beta = 2$ (b) $\alpha = -1, \beta = -2$ (c) $\alpha = -2, \beta = 2$ (d) $\alpha = -3, \beta = \dfrac{2}{3}$

Ans. (a)

23. The PDE $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y^2} = 0$ is

(a) hyperbolic for $x > 0,\ y < 0$ (b) elliptic for $x > 0,\ y < 0$

(c) hyperbolic for $x > 0,\ y > 0$ (d) elliptic for $x < 0,\ y > 0$

Ans. (a)

24. Let $u(x,t)$ satisfy the initial boundary value problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2};\ x\in(0,1),t>0$$

$$u(x,0)=\sin(\pi x);x\in[0,1]$$

$u(0,t)=u(1,t)=0,\ t>0$, Then for $x\in(0,1),u\left(x,\frac{1}{\pi^2}\right)$ is equal to

(a) $e\sin(\pi x)$ (b) $e^{-1}\sin(\pi x)$ (c) $\sin(\pi x)$ (d) $\sin(\pi^{-1}x)$

Ans. **(b)**

25. The values of α and β, such that

$$x_{n+1}=\alpha x_n\left(3-\frac{x_n^2}{a}\right)+\beta x_n\left(1+\frac{a}{x_n^2}\right)$$

has 3rd order convergence to $\sqrt{a}$, are

(a) $\alpha=\frac{3}{8},\beta=\frac{1}{8}$ (b) $\alpha=\frac{1}{8},\beta=\frac{3}{8}$ (c) $\alpha=\frac{2}{8},\beta=\frac{2}{8}$ (d) $\alpha=\frac{1}{4},\beta=\frac{3}{4}$

Ans. **(b)**

26. If $J[y]=\int_1^2\left(y'^2+2yy'+y^2\right)dx,\ y(1)=1$ and $y(2)$ is arbitrary then the extremal is

(a) e^{x-1} (b) e^{x+1} (c) e^{1-x} (d) e^{-x-1}

Ans. **(c)**

27. Let ϕ satisfy $\phi(x)=f(x)+\int_0^x\sin(x-t)\phi(t)\,dt$ .Then ϕ is given by

(a) $\phi(x)=f(x)+\int_0^x(x-t)f(t)\,dt$ (b) $\phi(x)=f(x)-\int_0^x(x-t)f(t)\,dt$

(c) $\phi(x)=f(x)-\int_0^x\cos(x-t)f(t)\,dt$ (d) $\phi(x)=f(x)-\int_0^x\sin(x-t)f(t)\,dt$

Ans. **(a)**

28. A bead slides without friction on a frictionless wire in the shape of a cycloid with equation

$x=a(\theta-\sin\theta),y=a(1+\cos\theta),0\le\theta\le 2\pi$. Then the Lagrangian function is

(a) $ma^2(1+\cos\theta)\theta^2-mga(1+\cos\theta)$ (b) $ma^2(1-\cos\theta)\theta^2-mga(1+\cos\theta)$

(c) $ma^2(1-\cos\theta)\theta^2+mga(1+\cos\theta)$ (d) $ma^2(1+\cos\theta)\theta^2-mga(1-\cos\theta)$

Ans. **(b)**

29. There are two boxes. Box 1 contains 2 red balls and 4 green balls. Box 2 contains 4 red balls and 2 green balls. A box is selected at random and a ball is chosen randomly from the selected box. If the ball turns out to be red, what is the probability that Box 1 had been selected?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{6}$

Ans. (b)

30. For any two events A and B, which of the following relations always holds ?

(a) $P^2\left(A\cap B^C\right)+P^2\left(A\cap B\right)+P^2\left(A^C\right)\geq\frac{1}{3}$

(b) $P^2\left(A\cap B^C\right)+P^2\left(A\cap B\right)+P^2\left(A^C\right)=\frac{1}{3}$

(c) $P^2\left(A\cap B^C\right)+P^2\left(A\cap B\right)+P^2\left(A^C\right)=1$

(d) $P^2\left(A\cap B^C\right)+P^2\left(A\cap B\right)+P^2\left(A^C\right)\leq\frac{1}{3}$

Ans. (a)

31. Suppose customers arrive in a shop according to a Poisson process with rate 4 per hour. The shop opens at 10:00 am. If it is given that the second customer arrives at 10:40 am, what is the probability that no customer arrived before 10:30 am ?

(a) $\frac{1}{4}$ (b) e^{-2} (c) $\frac{1}{2}$ (d) $e^{-1/2}$

Ans. (a)

32. Suppose $X_1, X_2, \ldots X_n$ is a random sample from a distribution with probability density function

$f(x)=3x^2 I_{(0,1)}(x)$, where $I_{(0,1)}(z)=\begin{cases}1 & \text{if } z\in(0,1)\\ 0 & \text{otherwise}\end{cases}$.

What is the probability density function $g(y)$ of $Y=\min\{X_1, X_2, \ldots X_n\}$?

(a) $g(y)=3ny^{3n-1}I_{(0,1)}(y)$

(b) $g(y)=1-\left(1-y^3\right)^n I_{(0,1)}(y)$

(c) $g(y)=\left(1-y^3\right)^n I_{(0,1)}(y)$

(d) $g(y)=3ny^2\left(1-y^3\right)^{n-1} I_{(0,1)}(y)$

Ans. (d)

33. $X_1, X_2, \ldots X_n$ are independent and identically distributed $N(\theta,1)$ random variables, where θ takes only integer values i.e. $\theta\in\{\ldots,-2,-1,0,1,2,\ldots\}$.

Which of the following is the maximum likelihood estimator of θ ?

(a) $\overline{X}$

(b) Integer closest to $\overline{X}$

(c) Integer part of $\overline{X}$ (Largest integer $\leq \overline{X}$)

(d) median of $\left(X_1, X_2, \ldots, X_n\right)$

Ans. (b)

34. ?

Ans. (a)

35. ?

Ans. (a)

36. ?

Ans. **(c)**

37. ?

Ans. **(c)**

38. ?

Ans. **(b)**

39. ?

Ans. **(d)**

40. **?**

Ans. **(a)**

PART-C

41. Let A be a n×n non-singular matrix with real entries. Let $B = A^T$ denote the transpose of A. Which of the following matrices are positive definite?

(a) A+B (b) $A^{-1} + B^{-1}$ (c) AB (d) ABA

Ans. **(c)**

42. Let $s \in (0,1)$. Then decide which of the following are true.

(a) $\forall m \in \mathbb{N},\ \ni n \in \mathbb{N} s.t \quad s > m/n$ (b) $\forall m \in \mathbb{N},\ \ni n \in \mathbb{N} s.t \quad s < m/n$

(c) $\forall m \in \mathbb{N},\ \ni n \in \mathbb{N} s.t \quad s = m/n$ (d) $\forall m \in \mathbb{N},\ \ni n \in \mathbb{N} s.t \quad s = m+n$

Ans. **(a, b)**

43. Let $f_n(x) = (-x)^n,\ x \in [0,1]$. Then decide which of the following are true.

(a) there exists a pointwise convergent subsequence of f_n

(b) f_n has no pointwise convergent subsequence

(c) f_n converges pointwise everywhere

(d) f_n has exactly one pointwise convergent subsequence

Ans. **(a)**

44. Which of the following are true for the function

$f(x) = \sin(x)\sin\left(\frac{1}{x}\right), x \in (0,1)$?

(a) $\underline{\lim}_{x\to 0} f(x) = \overline{\lim}_{x\to 0} f(x)$ (b) $\underline{\lim}_{x\to 0} f(x) < \overline{\lim}_{x\to 0} f(x)$

(c) $\underline{\lim}_{x\to 0} f(x) = 1$ (d) $\overline{\lim}_{x\to 0} f(x) = 0$

Ans. **(a, d)**

45. Find out which of the following series converge uniformly for $x \in (-\pi, \pi)$

(a) $\sum_{n=1}^{\infty} \frac{e^{-n|x|}}{n^3}$ (b) $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n^5}$ (c) $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$ (d) $\sum_{n=1}^{\infty} \frac{1}{((x+\pi)n)^2}$

Ans. **(a, b, c)**

46. Decide which of the following functions are uniformly continuous on $(0,1)$.

(a) $f(x)=e^x$ (b) $f(x)=x$ (c) $f(x)=\tan\left(\frac{\pi x}{2}\right)$ (d) $f(x)=\sin(x)$

Ans. **(a, b, d)**

47. Let $\chi_A(x)$ denote the function which is 1 if $x\in A$ and 0 otherwise. Consider

$$f(x)=\sum_{n=1}^{200}\frac{1}{n^6}\chi_{\left[0,\frac{n}{200}\right]}(x),\ x\in[0,1]$$

Then $f(x)$ is

(a) Riemann integrable on [0, 1]

(b) Lebesgue integrable on [0, 1]

(c) is a continuous function on [0, 1]

(d) is a monotone function on [0, 1]

Ans. **(a, b, d)**

48. A function $f(x,y)$ on $\mathbb{R}^2$ has the following partial derivatives

$$\frac{\partial f}{\partial x}(x,y)=x^2,\ \frac{\partial f}{\partial y}(x,y)=y^2$$

Then

(a) f has directional derivatives in all directions everywhere.

(b) f has a derivative at all points.

(c) f has directional derivative only along the direction (1,1) everywhere.

(d) f does not have directional derivatives in any direction everywhere

Ans. **(a, b)**

49. Let d_1, d_2 be the following metrics on $\mathbb{R}^n$.

$$d_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|,\ d_2(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{1/2}$$

Then decide which of the following is a metric on $\mathbb{R}^n$.

(a) $d(x,y)=\dfrac{d_1(x,y)+d_2(x,y)}{1+d_1(x,y)+d_2(x,y)}$

(b) $d(x,y)=d_1(x,y)-d_2(x,y)$

(c) $d(x,y)=d_1(x,y)+d_2(x,y)$

(d) $d(x,y)=e^{\pi}d_1(x,y)+e^{-\pi}d_2(x,y)$

Ans. **(a, c, d)**

50. Let A be the following subset of $\mathbb{R}^2$.

$$A=\left\{(x,y):(x+1)^2+y^2\le 1\right\}\cup\left\{(x,y):y=x\sin\frac{1}{x},x>0\right\}$$

Then

(a) A is connected

(b) A is compact

(c) A is path connected

(d) A is bounded

Ans. **(a, c)**

51. Let

$$\ell^{\infty} = \left\{ \underline{a} = (a_k)_{k\geq 1} : a_k \in \mathbb{C}, \sup_k |a_k| \equiv \|\underline{a}\| < \infty \right\}$$

$$\ell^{2} = \left\{ \underline{a} = (a_k)_{k\geq 1} : a_k \in \mathbb{C} \left(\Sigma |a_k|^2 \right)^{1/2} \equiv \|\underline{a}\|_2 < \infty \right\}$$

$T : \ell^{\infty} \to \ell^2$ as

Define a map

$$T\underline{a} = \left\{ a_1, \frac{a_2}{2}, \frac{a_3}{3}, ... \right\}$$

Which of the following statements is true?

(a) T is a continuous linear map

(b) T maps ℓ^{∞} onto ℓ^2

(c) T^{-1} exists and is continuous

(d) T is uniformly continuous

Ans. (a, d)

52. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that a_{ij} is an integer for all $i. j$, Let $AB = I$ with $B = [b_{ij}]$ (where I is the identity matrix). For a square matrix C, det C denotes its determinant. Which of the following statements is true?

(a) If det A = 1 then det B = 1

(b) A sufficient condition for each b_{ij} to be an integer is that det A is an integer

(b) B is always an integer matrix

(d) A necessary condition for each b_{ij} to be an integer is $\det A \in \{-1, +1\}$

Ans. (a, d)

53. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and let α_n and β_b denote the two eigenvalues of A^n such that $|\alpha_n| \geq |\beta_n|$. Then

(a) $a_n \to \infty$ as $n \to \infty$

(c) $\beta_n \to 0$ as $n \to \infty$

(c) β_n is positive if n is even.

(d) β_n is negative if n is odd

Ans. (a, b, c, d)

54. Let M_n denote the vector space of all $n \times n$ real matrices. Among the following subsets of M_n, decide which are linear subspaces.

(a) $V_1 = \{A \in M_n : A \text{ is nonsingular}\}$

(b) $V_2 = \{A \in M_n : \det(A) = 0\}$

(c) $V_3 = \{A \in M_n : \text{trace}(A) = 0\}$

(d) $V_4 = \{BA : A \in M_n\}$ where B is some fixed matrix in M_n

Ans. (c, d)

55. If P are Q invertible matrices such that PQ = – QP, then we can conclude that

(a) $Tr(P) = Tr(Q) = 0$

(b) $Tr(P) = Tr(Q) = 1$

(c) $Tr(P) = -Tr(Q)$

(c) $Tr(P) \neq Tr(Q)$

Ans. (a, c)

56. Let n be an odd number ≥ 7. Let $A=\left[a_{ij}\right]$ be an $n\times n$ matrix with $a_{i,i+1}=1$ for all $i=1,2,....,n-1$ and $a_{n,1}=1$. Let $a_{ij}=0$ for all the other pairs (i,j). Then we can conclude that

(a) A has 1 as an eigenvalue.

(b) A has –1 as an eigenvalue.

(c) A has at least one eigenvalue with multiplicity ≥ 2.

(d) A has no real eigenvalues

Ans. **(a)**

57. Let W_1, W_2, W_3 be three distinct subspaces of $\mathbb{R}^{10}$ such that each W_i has dimension 9. Let $W=W_1\cap W_2\cap W_3$. Then we can conclude that

(a) W may not be a subspace of $\mathbb{R}^{10}$

(b) $\dim W\leq 8$

(c) $\dim W\geq 7$

(d) $\dim W\leq 3$

Ans. **(b, c)**

58. Let A be a real symmetric matrix. Then we can conclude that

(a) A does not have 0 as an eigenvalue

(b) All eigenvalues of A are real

(c) If A^{-1} exists, then A^{-1} is real and symmetric

(d) A has at least one positive eigenvalue

Ans. **(b, c)**

59. Let $f(z)$ be the meromorphic function given by $\dfrac{z}{\left(1-e^z\right)\sin z}$. Then

(a) z = 0 is a pole of order 2.

(b) for every $k\in\mathbb{Z}, z=2\pi ik$ is a simple pole.

(c) for every $k\in\mathbb{Z}\setminus\{0\}$, $z=k\pi$ is a simple pole

(d) $z=\pi+2\pi i$ is a pole

Ans. **(b, c)**

60. Consider the polynomial $P(z)=\sum_{n=1}^{N}a_nz^n, 1\leq N<\infty, a_n\in\mathbb{R}\setminus\{0\}$. Then with $\mathbb{D}=\{w\in\mathbb{C}:|w|<1\}$

(a) $P(\mathbb{D})\subseteq\mathbb{R}$

(b) $P(\mathbb{D})$ is open

(c) $P(\mathbb{D})$ is closed

(d) $P(\mathbb{D})$ is bounded

Ans. **(b, d)**

61. Consider the polynomial $P(z)=\left(\sum_{n=0}^{5}a_nz^n\right)\left(\sum_{n=0}^{9}b_nz^n\right)$ where $a_n, b_n\in\mathbb{R}\ \forall n, a_5\neq 0, b_9\neq 0$. Then counting roots with multiplicity we can conclude that $P(z)$ has

(a) at least two real roots

(b) 14 complex roots

(c) no real roots

(d) 12 complex roots

Ans. **(a, b) or (a, b, d)**

62. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Let $g:\mathbb{D}\to\mathbb{D}$ be holomorphic, $g(0)=0$, and

let. $h(z)=\begin{cases} g(z)/z, & z\in\mathbb{D}, z\neq 0 \\ g'(0), & z=0\end{cases}$

Which of the following statements are true?

(a) h is holomorphic in $\mathbb{D}$.

(b) $h(\mathbb{D})\subseteq\overline{\mathbb{D}}$

(c) $|g'(0)|>1$

(d) $|g(1/2)|\leq 1/2$

Ans. **(a, b, d)**

63. Consider the following subsets of the group of 2×2 non-singular matrices over $\mathbb{R}$:

$$G=\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}: a,b,d\in\mathbb{R},\ ad=1\right\}$$

$$H=\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}: b\in\mathbb{R}\right\}$$

Which of the following statements are correct?

(a) G forms a group under matrix multiplication.

(b) H is a normal subgroup of G.

(c) The quotient group G/H is well-defined and is Abelian.

(d) The quotient group G/H is well defined and is isomorphic to the group of 2×2 diagonal matrices (over $\mathbb{R}$) with determinant 1

Ans. (a, b, c, d)

64. Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^*$ be the group of non zero complex numbers under multiplication. Then which of the following are true ?

(a) $\mathbb{C}^*$ is cyclic

(b) Every finite subgroup of $\mathbb{C}^*$ is cyclic

(c) $\mathbb{C}^*$ has finitely many finite subgroups

(d) Every proper subgroup of $\mathbb{C}^*$ is cyclic

Ans. (b)

65. Let R be a finite non-zero commutative ring with unity. Then which of the following statements are necessarily true?

(a) Any non-zero element of R is either a unit or a zero divisor.

(b) There may exist a non-zero element of R which is neither a unit nor a zero divisor.

(c) Every prime ideal of R is maximal.

(d) If R has no zero divisors then order of any additive subgroup of R is a prime power.

Ans. (a,c,d)

66. Which of the following statements are true?

(a) $\mathbb{Z}[x]$ is a principal ideal domain.

(b) $\mathbb{Z}[x,y]/\langle y+1\rangle$ is a unique factorization domain.

(c) If is a principal ideal domain and *p* is a non-zero prime ideal, then *R/p* has finitely many prime ideals

(d) If *R* is a principal ideal domain, then any subring of *R* containing 1 is again a principal ideal domain

Ans. (b, c)

67. Let R be a commutative ring with unity and $R[x]$ be the polynomial ring in one variable. For a non zero $f=\Sigma_{n=0}^{N}a_n x^n$, define $\omega(f)$ to be the smallest n such that $a_n\neq 0$. Also $\omega(0)=+\infty$.

Then which of the following statements is/are true ?

(a) $\omega(f+g)\geq\min(\omega(f),\omega(g))$

(b) $\omega(fg)\geq\omega(f)+\omega(g)$

(c) $\omega(f+g)=\min(\omega(f),\omega(g))$ if $\omega(f)\neq\omega(g)$

(d) $\omega(fg)=\omega(f)+\omega(g)$ if R is an integral domain

Ans. (a, b, c, d)

68. Let $\mathbb{F}_2$ be the finite field of order 2. Then which of the following statements are true ?

(a) $\mathbb{F}_2[x]$ has only finitely many irreducible elements.

(b) $\mathbb{F}_2[x]$ has exactly one irreducible polynomial of degree 2.

(c) $\mathbb{F}_2[x]/\langle x^2+1\rangle$ is a finite dimensional vector space over $\mathbb{F}_2$.

(d) Any irreducible polynomial in $\mathbb{F}_2[x]$ of degree 5 has distinct roots in any algebraic closure of $\mathbb{F}_2$

Ans. **(b, c, d)**

69. Let (X,d) be a metric space. Then

(a) An arbitrary open set G in X is a countable union of closed sets.

(b) An arbitrary open set G in X cannot be countable union of closed sets if X is connected.

(c) An arbitrary open set G in X is a countable union of closed sets only if X is countable.

(d) An arbitrary open set G in X is a countable union of closed sets only if X is locally compact

Ans. **(a)**

70. Let $S=\{x, y\in\mathbb{R}^2 \mid -1\le x\le 1 \text{ and } -1\le y\le 1\}$

Let $T=S\setminus(0,0)$, the set obtained by removing the origin from S.

Let f be a continuous function from T to $\mathbb{R}$. Choose all correct options.

(a) Image of f must be connected.

(b) Image of f must be compact.

(c) Any such continuous function f can be extended to a continuous function from S to $\mathbb{R}$.

(d) If f can be extended to a continuous function from S to $\mathbb{R}$ then the image of f is bounded

Ans. **(a, d)**

71. Let $x:[0,3\pi]\to\mathbb{R}$ be a nonzero solution of the ODE

$$x''(t)+e^{t^2}x(t)=0, \text{ for } t\in[0,3\pi].$$

Then the cardinality of the set $\{t\in[0,3\pi]: x(t)=0\}$ is

(a) equal to 1

(b) greater than or equal to 2

(c) equal to 2

(d) greater than or equal to 3

Ans. **(b, d)**

72. Consider the initial value problem

$$y'(t)=f(y(t)), \qquad y(0)=a\in\mathbb{R} \text{ where } f:\mathbb{R}\to\mathbb{R}.$$

Which of the following statements are necessarily true ?

(a) There exists a continuous function $f:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$ such that the above problem does not have a solution in any neighbourhood of 0.

(b) The problem has a unique solution for every $a\in\mathbb{R}$ when f is Lipschitz continuous.

(c) When f is twice continuously differentiable, the maximal interval of existence for the above initial value problem is $\mathbb{R}$.

(d) The maximal interval of existence for the above problem is $\mathbb{R}$ when f is bounded and continuously differentiable

Ans. **(b, d)**

73. Let $(x(t), y(t))$ satisfy for $t > 0$

$$\frac{dx}{dt} = -x + y, \frac{dy}{dt} = -y, \quad x(0) = y(0) = 1$$

Then $x(t)$ is equal to

(a) $e^{-t} + t \quad y(t)$ (b) $y(t)$ (c) $e^{-t}(1+t)$ (d) $-y(t)$

Ans. (a, c)

74. Consider the wave equation for $u(x,t)$

$$\left.\begin{array}{ll} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x^2} = 0, & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x), & x \in \mathbb{R} \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & x \in \mathbb{R} \end{array}\right\}$$

Let u_i be the solution of the above problem with $f = f_i$ and $g = g_i$ for $i = 1, 2$, where $f_i : \mathbb{R} \to \mathbb{R}$ and $g_i : \mathbb{R} \to \mathbb{R}$ are given C^2 functions satisfying $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, for every $x \in [-1,1]$. Which of the following statements are necessarily true

(a) $u_1(0,1) = u_2(0,1)$ (b) $u_1(1,1) = u_2(1,1)$

(c) $u_1\left(\frac{1}{2}, \frac{1}{2}\right) = u_2\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $u_1(0,2) = u_2(0,2)$

Ans. (a, c)

75. Let $u : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be a C^2 function satisfying $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, for all $(x,y) \neq (0,0)$. Suppose u is of the form $u(x,y) = f\left(\sqrt{x^2 + y^2}\right)$, where $f : (0, \infty) \to \mathbb{R}$, is a nonconstant function, then

(a) $\lim_{x^2+y^2 \to 0} |u(x,y)| = \infty$ (b) $\lim_{x^2+y^2 \to 0} |u(x,y)| = 0$

(c) $\lim_{x^2+y^2 \to \infty} |u(x,y)| = \infty$ (d) $\lim_{x^2+y^2 \to \infty} |u(x,y)| = 0$

Ans. (a, c)

76. The Cauchy problem

$$\left.\begin{array}{r} y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0 \\ u = g \text{ on } \Gamma \end{array}\right\}$$

has a unique solution in a neighbourhood of Γ for every differentiable function $g : \Gamma \to \mathbb{R}$ if

(a) $\Gamma = \{(x,0) : x > 0\}$ (b) $\Gamma = \{(x,y) : x^2 + y^2 = 1\}$

(c) $\Gamma = \{(x,y) : x + y = 1, x > 1\}$ (d) $\Gamma = \{(x,y) : y = x^2, \ x > 0\}$

Ans. (a, c, d)

77. The order of linear multi step method $u_{j+1} = (1-a)u'_j + au_{j-1} + \frac{h}{4}\{(a+3)u'_{j+1} + 3a + 1uj - 1'$ for solving $u' = f(x,u)$ is

(a) 2 if $a = -1$ (b) 2 if $a = -2$ (c) 3 if $a = -1$ (d) 3 if $a = -2$

Ans. **(b, c)**

78. The functional $J[y] = \int_0^1 (y'^2 + x^2)\,dx$ where $y(0) = -1$ and $y(1) = 1$ on $y = 2x - 1$, has

(a) weak minimum (b) weak maximum

(c) strong minimum (d) strong maximum

Ans. **(a, c) or (c)**

79. Let $y(x)$ be a piecewise continuously differentiable function on [0,4]. Then the functional

$J[y] = \int_0^4 (y'-1)^2 (y'+1)^2\,dx$ attains minimum if $y = y(x)$ is

(a) $y = \frac{x}{2} \quad 0 \le x \le 4$ (b) $y = \begin{cases} -x & 0 \le x \le 1 \\ x-2 & 1 \le x \le 4 \end{cases}$

(c) $y = \begin{cases} 2x & 0 \le x \le 2 \\ -x+6 & 2 \le x \le 4 \end{cases}$ (d) $y = \begin{cases} x & 0 \le x \le 3 \\ -x+6 & 3 \le x \le 4 \end{cases}$

Ans. **(b, d)**

80. Which of the following are the characteristic numbers and the corresponding eigenfunctions for the Fredholm homogeneous equation whose kernel is

$K(x,t) = \begin{cases} (x+1)t, & 0 \le x \le t \\ (t+1)x, & t \le x \le 1 \end{cases}$?

(a) $1, e^x$ (b) $-\pi^2, \pi \sin \pi x + \cos \pi x$

(c) $-4\pi^2, \pi \sin \pi x + \pi \cos 2\pi x$ (d) $-\pi^2, \pi \cos \pi x + \sin \pi x$

Ans. **(a, d)**

81. The integral equation $\phi(x) - \frac{2}{\pi}\int_0^\pi \cos(x+t)\phi(t)\,dt = f(x)$ has infinitely many solutions if

(a) $f(x) = \cos x$ (b) $f(x) = \cos 3x$ (c) $f(x) = \sin x$ (d) $f(x) = \sin 3x$

Ans. **(b, c, d)**

82. Which of the following are canonical transformations? (Where q,p represent generalized coordinate and generalised momentum respectively)

(a) $P = \log \sin p, \quad Q = q \tan p$ (b) $P = qp^2, Q = \frac{1}{p}$

(c) $P = q \cot p, \quad Q = \log\left(\frac{1}{q}\sin p\right)$ (d) $P = q^2 \sin 2p, \quad Q = q^2 \cos 2p$

Ans. **(a, b, c)**