CSIR-NET-(P YQ) MATHEMATICAL SCIENCE
JUNE-2024

SECTION-A

1. Consider the set $A=\left\{x\in\mathbb{Q}:0<\left(\sqrt{2}-1\right)x<\sqrt{2}+1\right\}$ as a subset of $\mathbb{R}$. Which of the following statements is true?

(a) sup $A=2+2\sqrt{3}$ (b) sup $A=3+2\sqrt{2}$

(c) inf $A=2+2\sqrt{3}$ (d) inf $A=3+2\sqrt{2}$

Ans. (b)

2. Let $S=\left\{x\in\mathbb{R}:x>1\text{ and }\frac{1-x^4}{1-x^3}\right\}$. Which of the following is true about S.

(a) S is empty.

(b) There is a bijection between S and $\mathbb{N}$.

(c) There is a bijection between S and $\mathbb{R}$

(d) There is a bijection between S and a non-empty finite set

Ans. (c)

3. Let C be the collection of all sets S such that the power set of S is countably infinite. Which of the following statements is true?

(a) There exists a non-empty finite set in C

(b) There exists a countably infinite set in C

(d) There exists an uncountable set in C

(d) C is empty

Ans. (d)

4. Let $(a_n)_{a\geq1}$ be a bounded sequence in $\mathbb{R}$. Which of the following statements is FALSE?

(a) if $\lim_{n\to\infty}\inf a_n=\lim_{n\to\infty}\sup a_n$, then (a_n) is convergent

(b) if $\inf\left\{a_n\mid n\geq1\right\}=\lim_{n\to\infty}\sup a_n$ then (a_n) is convergent

(c) if $\sup\left\{a_n\mid n\geq1\right\}=\lim_{n\to\infty}\inf a_n$, then (a_n) is constant

(d) if $\sup\left\{a_n\mid n\geq1\right\}=\inf\left\{a_n\mid n\geq1\right\}$, then (a_n) is constant

Ans. (c)

5. What is the cardinality of the set of real solutions of $e^x+x=1$?

(a) 0 (b) 1

(c) Countably infinite (d) Uncountable

Ans. (b)

6. . For each $n \geq 1$ define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{x^2}{\sqrt{x^2 + \frac{1}{n}}}, x \in \mathbb{R}$$

where $\sqrt{\ }$ denotes the non-negative square root. Wherever $\lim_{n \to \infty} f_n(x)$, denote it by $f(x)$.

Which of the following statements is true ?

(a) There exists $x \in \mathbb{R}$ such that $f(x)$ is not defined

(b) $f(x) = 0$ for all $x \in \mathbb{R}$

(c) $f(x) = x$ for all $x \in \mathbb{R}$

(d) $f(x) = |x|$ for all $x \in \mathbb{R}$

Ans. **(d)**

7. Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a non-zero linear transformation. Which of the following statements is true?

(a) If A is one-to-one but not onto, then $m > n$

(b) If A is onto but not one-to-one, then $m < n$

(c) If A is bijective, then $m = n$

(c) If A is one-to-one, then $m = n$

Ans. **(c)**

8. Let A be a 10×10 real matrix. Assume that the rank of A is 7. Which of the following statements is necessarily true ?

(a) There exists a vector $v \in \mathbb{R}^{10}$ such that $Av \neq 0$ and $A^2 v = 0$

(b) There exists a vector $v \in \mathbb{R}^{10}$ such that $A^2 v \neq 0$

(c) A must have a non-zero eigenvalue

(d) $A^7 = 0$

Ans. **(b)**

9. Let $\begin{pmatrix} 2 & a \\ b & c \end{pmatrix}$ be a 2×2 real matrix for which 6 is an eigenvalue. Which of the following statements is necessarily true ?

(a) $24 - ab = 4c$ (b) $a + b = 8$ (c) $c = 6$ (d) $ab = 0$

Ans. **(a)**

10. Let v be the real vector space of 2×2 matrices with entries in $\mathbb{R}$. Let $T : V \to V$ denote the linear transformation defined by $T(B) = AB$ for all $B \in V$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. What is the characteristic polynomial of T ?

(a) $(x-2)(x-1)$ (b) $x^2(x-2)(x-1)$ (c) $(x-2)^2(x-1)^2$ (d) $(x^2-2)(x^2-1)$

Ans. **(c)**

11. Let $A=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$, and consider the symmetric bilinear form on $\mathbb{R}^4$ given by $\langle v,w\rangle = v^t\ Aw$, for $v,w\in\mathbb{R}^4$. Which of the following statements is true ?

(a) A is invertible

(b) There exist non-zero vectors *v,w* such that $\langle v,w\rangle = 0$

(c) $\langle u,v\rangle \neq (u,w)$ for all non-zero vectors u, v, w with $v\neq w$

(d) Every eigenvalue of A^2 is positive

Ans. (b)

12. For a quadratic form $f(x,y,z)\in\mathbb{R}[x,y,z]$, we say that $(a,b,c)\in\mathbb{R}^3$ is a zero of *f* if $f(a,b,c)=0$. Which of the following quadratic forms has at least one zero different from (0,0,0) ?

(a) $x^2+2y^2+3z^2$

(b) $x^2+2y^2+3z^2-2xy$

(c) $x^2+2y^2+3z^2-2xy-2yz$

(d) $x^2+2y^2-3z^2$

Ans. (d)

13. Let *f* be an entire function. Which of the following statements is FALSE?

(a) If $Re(f)$, $Im(f)$ are bounded then *f* is constant

(b) If $e^{|Re(f)|+|Im(f)|}$ is bounded, then *f* is constant

(c) If the sum $Re(f)+Im(f)$ and the product $Re(f)\,Im(f)$ are bounded, then *f* is constant

(d) If $\sin(Re(f)+Im(f))$ is bounded, then *f* is constant

Ans. (d)

14. Consider the contour γ given by $\gamma(\theta)=\begin{cases}e^{2i\theta} & \text{for } \theta\in[0,\pi/2]\\ 1+2e^{2i\theta} & \text{for } \theta\in[\pi/2,3\pi/2]\\ e^{2i\theta} & \text{for } \theta\in[3\pi/2,2\pi]\end{cases}$. Then what is the value of $\int_\gamma \frac{dz}{z(z-2)}$?

(a) 0

(b) πi

(c) $-\pi i$

(d) $2\pi i$

Ans. (c)

15. Let *a, b* be two real numbers such that $a<0<b$. For a positive real number *r*, define $\gamma_r(t)=re^{it}$ (where $t\in[0,2\pi]$) and $I_r=\frac{1}{2\pi i}\int_{\gamma_r}\frac{z^2+1}{(z-a)(z-b)}dz$. Which of the following statements is necessarily true?

(a) $I_r\neq 0$ if $r>\max\{|a|,b\}$

(b) $I_r\neq 0$ if $r<\max\{|a|,b\}$

(c) $I_r=0$ if $r>\max\{|a|,b\}$ and $|a|=b$

(d) $I_r=0$ if $|a|<r<b$

Ans. (c)

16. For a complex number *a* such that $0<|a|<1$, which of the following statements is true ?

(a) If $|z|<1$, then $|1-\bar{a}z|<|z-a|$

(b) If $|z-a|=|1-\bar{a}z|$, then $|z|=1$

(c) If $|z|=1$, then $|z-a|<|1-\bar{a}z|$

(d) If $|1-\bar{a}z|<|z-a|$, then $|z|<1$

Ans. (b)

17. How many arrangements of the digits of the number 1234567 are there, such that exactly three of them occur in their original position. (E.g., in the arrangement 5214763, exactly the digits 2,4 and 6 are in their original positions. In the arrangement 1243576, exactly the digits 1, 2 and 5 are in their original positions.)

(a) 525 (b) 35 (c) 840 (d) 315

Ans. (d)

18. The number of group homomorphisms from $\mathbb{Z}/150\mathbb{Z}$ to $\mathbb{Z}/90\mathbb{Z}$ is

(a) 30 (b) 60 (c) 45 (d) 10

Ans. (a)

19. Consider the ring $R = \{\sum_{n\in\mathbb{Z}} a_n X^n \mid a_n \in \mathbb{Z}$; and $a_n \neq 0$ only for finitely many $n \in \mathbb{Z}\}$ where addition and multiplication are given by

$$\sum_{n\in\mathbb{Z}} a_n X^n + \sum_{n\in\mathbb{Z}} b_n X^n = \sum_{n\in\mathbb{Z}} (a_n + b_n) X^n$$

$$\left(\sum_{n\in\mathbb{Z}} a_n X^n\right)\left(\sum_{m\in\mathbb{Z}} b_m X^m\right) = \sum_{k\in\mathbb{Z}}\left(\sum_{n+m=k} a_n b_m\right) X^k$$

Which of the following statements is true?

(a) R is not commutative

(b) The ideal $(X-1)$ is a maximal ideal in R

(c) The ideal $(X-1, 2)$ is a prime ideal in R

(d) The ideal $(X, 5)$ is a maximal ideal in R

Ans. (c)

20. Let S be a dense subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ given function. Define $g : S \to \mathbb{R}$ by $g(x) = f(x)$.

Which of the following statements is necessarily true ?

(a) If f is continuous on the set S, then f is continuous on the set $\mathbb{R}\backslash S$

(b) If g is continuous, then f is continuous on the set S

(c) If g is identically 0 and f is continuous on the set $\mathbb{R}\backslash S$, then f is identically 0

(d) If g is identically 0 and f is continuous on the set S, then f is identically 0

Ans. (d)

21. Consider the initial value problem (IVP)

$$\begin{cases} y'(x) = \sqrt{|y(x) + \in|}, x \in \mathbb{R} \\ y(0) = y_0 \end{cases}$$

Consider the following statements:

S1: There is an $\in > 0$ such that for all $y_0 \in \mathbb{R}$, , the IVP has more than one solution.

S2: There is a $y_0 \in \mathbb{R}$ such that for all $\in > 0$, the IVP has more than one solution. Then

(a) both S_1 and S_2 are true (b) S_1 is true but S_2 is false

(c) S_1 is false but S2 is true (d) both S_1 and S_2 are false

Ans. (d)

22. Let φ denote the solution to the boundary value problem (BVP)

$$\begin{cases} (xy')' - 2y' + \dfrac{y}{x} = 1, & 1 < x < e^4 \\ y(1) = 0, & y(e^4) = 4e^4 \end{cases}$$

Then the value of $\varphi(e)$ is

(a) $-\dfrac{e}{2}$ (b) $-\dfrac{e}{3}$ (c) $\dfrac{e}{3}$ (d) e

Ans. (a)

23. Let $u = u(x, t)$ be the solution of the following initial value problem

$$\begin{cases} u_t + 2024u_x = 0 & , x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x) & , x \in \mathbb{R} \end{cases}$$

Where $u_0 : \mathbb{R} \to \mathbb{R}$ arbitrary C^1 function. Consider the following statements

S_1: If $A_t := \{x \in \mathbb{R} : u(x,t) < 1\}$ and $|A_t|$ denotes the Lebesgue measure of A_t for every $t \geq 0$, then $|A_t| = |A_0|, \forall\, t > 0$.

S_2: If u_0 is Lebesgue integrable, then for every $t > 0$, the function $x \mapsto u(x,t)$ is Lebesgue integrable.

Then

(a) both S_1 and S_2 are true (b) S_1 is true but S_2 is false

(c) S_2 is true but S_1 is false (d) both S_1 and S_2 are false

Ans. (a)

24. If $u = u(x,t)$ is the solution of the initial value problem

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin(4x) + x + 1 & x \in \mathbb{R} \end{cases}$$

satisfying $|u(x,t)| < 3e^{x^2}$ for all $x \in \mathbb{R}$ and $t > 0$, then

(a) $u\left(\dfrac{\pi}{8},1\right) + u\left(-\dfrac{\pi}{8},1\right) = 2$ (b) $u\left(\dfrac{\pi}{8},1\right) = u\left(-\dfrac{\pi}{8},1\right)$

(c) $u\left(\dfrac{\pi}{8},1\right) + 2u\left(-\dfrac{\pi}{8},1\right) = 2$ (d) $u\left(\dfrac{\pi}{8},1\right) = -u\left(-\dfrac{\pi}{8},1\right)$

Ans. (a)

25. If the value of the approximate solution of the initial value problem

$$\begin{cases} y'(x) = x(y(x)+1), & x \in \mathbb{R} \\ y(0) = \beta \end{cases}$$

at $x = 0.2$ using the forward Euler method with step size 0.1 is 1.02, then the value of β is

(a) 0 (b) -1 (c) 2 (d) 1

Ans. (d)

26. Let $B(0,1)=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$ be the open unit disc in $\mathbb{R}^2$, $\partial B(0,1)$ denote the boundary of $B(0,1)$, and ν denote unit outward normal to $\partial B(0,1)$. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a given continuous function.

The Euler-Lagrange equation of the minimization problem

$$\min\left\{\frac{1}{2}\iint_{B(0,1)}|\nabla u|^2\,dxdy+\frac{1}{2}\iint_{B(0,1)}e^{u^2}\,dxdy+\int_{\partial B(0,1)}fu\,ds\right\}\text{ subject to } u\in C^1\left(\overline{B(0,1)}\right)\text{ is}$$

(a) $\begin{cases}\Delta u=-ue^{u^2} & \text{in } B(0,1)\\ \dfrac{\partial u}{\partial \nu}=f & \text{on } \partial B(0,1)\end{cases}$

(b) $\begin{cases}\Delta u=ue^{u^2}+f & \text{in } B(0,1)\\ u=0 & \text{on } \partial B(0,1)\end{cases}$

(c) $\begin{cases}\Delta u=ue^{u^2} & \text{in } B(0,1)\\ \dfrac{\partial u}{\partial \nu}=-f & \text{on } \partial B(0,1)\end{cases}$

(d) $\begin{cases}\Delta u=ue^{u^2} & \text{in } B(0,1)\\ \dfrac{\partial u}{\partial \nu}+u=f & \text{on } \partial B(0,1)\end{cases}$

Ans. **(c)**

27. Let u be the solution of the Volterra integral equation $\int_0^t\left[\frac{1}{2}+\sin(t-\tau)\right]u(\tau)\,d\tau=\sin t$. Then the value of $u(1)$ is

(a) 0 (b) 1 (c) 2 (d) $2e^{-1}$

Ans. **(a)**

28. Consider a solid circular cylinder of radius 2 meters and height 3 meters of uniform density. If the density of the cylinder is ρ kg/meter^3, then the moment of inertia (in kg meter^2) of the cylinder about a diameter of its base is

(a) $48\pi\rho$ (b) $43\pi\rho$ (c) $24\pi\rho$ (d) $4\pi\rho$

Ans. **(a)**

29. Let A_1, A_2, A_3 be events satisfying $0<P(A_i)<1$ for $i=1,2,3$. Which of the following statements is true ?

(a) $P(A_1\mid A_2)P(A_2\mid A_3)\le P(A_1\mid A_3)$

(b) $P(A_1\mid A_2)P(A_3\mid A_2)\ge P(A_1\cap A_3\mid A_2)$

(c) $P(A_1\mid A_2)+P(A_3\mid A_2)\ge P(A_1\cup A_3\mid A_2)$

(d) $P(A_1\mid A_2)+P(A_2\mid A_3)\le P(A_1\mid A_3)$

Ans. **(c)**

30. Let X be a random variable with cumulative distribution function given by

$$F(x)=\begin{cases}0, & \text{if } x<0\\ \dfrac{x+1}{3}, & \text{if } 0\le x<1\\ 1, & \text{if } x\ge 1\end{cases}$$

Then the value of $P\left(\frac{1}{3}<X<\frac{3}{4}\right)+P(X=0)$ is equal to

(a) $\frac{7}{36}$ (b) $\frac{11}{36}$ (c) $\frac{13}{36}$ (d) $\frac{17}{36}$

Ans. **(d)**

31. Let $\{X_n \mid n \geq 0\}$ be a homogeneous Markov chain with state space $S = \{0,1,2,3,4\}$ and transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/4 & 0 & 0 & 3/4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 1/4 & 0 \\ 1/8 & 1/8 & 1/2 & 1/8 & 1/8 \end{pmatrix} \end{array}$$

Let α denote the probability that starting with state 4 the chain will eventually get absorbed in closed class $\{0,3\}$. Then the value of α is

(a) $\frac{6}{21}$ (b) $\frac{11}{21}$ (c) $\frac{8}{21}$ (d) $\frac{10}{21}$

Ans. **(d)**

32. Let a point P be chosen at random on the line segment AB of length α. Let Z_1 and Z_2 denote the lengths of line segments AP and BP respectively. Then the value of $E(|Z_1 - Z_2|)$ is

(a) α (b) 2α (c) $\frac{\alpha}{2}$ (d) $\frac{2\alpha}{3}$

Ans. **(c)**

33. Consider a distribution with probability mass function

$$f(x \mid \theta) = \begin{cases} \frac{1-\theta}{2} & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{\theta}{2} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

, where $\theta \in (0,1)$ is an unknown parameter. In a random sample of size 100 from the above distribution, the observed counts of 0, 1 and 2 are 20, 30 and 50 respectively. Then, the maximum likelihood estimate of θ based on the observed data is

(a) 1 (b) 5/7 (c) 1/2 (d) 2/7

Ans. **( )**

34. Let X_1, X_2 be a random sample from $N(0, \sigma^2)$ distribution, where $\sigma > 0$ and $N(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Suppose, for some constant, c, $\left(c\left(X_1^2 + X_2^2\right), \infty\right)$ is a confidence interval for variance σ^2 with confidence coefficient 0.95. Then the value of c is equal to

(a) $-2\ln(0.05)$ (b) $-2\ln(0.95)$

(c) $-\frac{1}{-2\ln(0.05)}$ (d) $-\frac{1}{-2\ln(0.95)}$

Ans. **( )**

35. Let X_1, X_2 be a random sample from a population having probability density function $f \in \{f_0, f_1\}$ where

$$f_0(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \text{ and } f_1(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$. For testingthe nullhypothesis $H_0 = f = f_0$

against the alternate hypothesis $H_1 = f = f_1$, the power of a most powerful test of size $\alpha = 0.05$ is equal to

(a) 0.4625 (b) 0.5425 (c) 0.7625 (d) 0.6225

Ans. **( )**

36. Let $X_1,\ldots,X_{10}$ be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1}, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta > 0$ is an unknown parameter. The prior distribution of θ is given by

$$\pi(\theta) = \begin{cases} \theta x^{-\theta}, & \text{if } \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

The Bayes estimator of θ under squared error loss is

(a) $\dfrac{12}{1-\sum_{i=1}^{10} \ln X_i}$ (b) $\dfrac{11}{2-\sum_{i=1}^{10} \ln X_i}$ (c) $\dfrac{3+\sum_{i=1}^{10} \ln X_i}{13}$ (d) $\dfrac{2+\sum_{i=1}^{10} \ln X_i}{111}$

Ans. **( )**

37. An analyst considers standardized values of observations on three variables, consumption (C), saving (S) and total income (TI) so that they have zero means and unit variances. She further considers disposable income (DI) where $DI = C+S$. In the simple linear regressions of DI on TI, DI on C and S on TI, the regression coefficients are 0.8, 0.5 and 0.4, respectively. There are 21 sample observations. Sample covariances and variances are calculated with divisor 20. Then, the value of sum of squared residuals in the regression of DI on S is

(a) 5 (b) 10 (c) 15 (d) 20

Ans. **( )**

38. Let $X_0, X_1,\ldots, X_p (p \ge 2)$ be independent and identically distributed random variables with mean 0 and variance 1. Suppose $Y_i = Y_0 + X_i$, $i = 1,\ldots, p$. The first principal component based on the covariance matrix of $\underline{Y} = (Y_1,\ldots Y_p)^T$ is

(a) $\dfrac{1}{\sqrt{p}}\sum_{i=1}^{p} Y_i$ (b) $\dfrac{1}{p}\sum_{i=1}^{p} Y_i$ (c) $\sqrt{p}\sum_{i=1}^{p} Y_i$ (d) $\sum_{i=1}^{p} Y_i$

Ans. **( )**

39. The expected number of distinct units in asimple random sample of 3 units drawn with replacement from a population of 100 units is

(a) $3-\left(\dfrac{99}{100}\right)^3$ (b) $100-\dfrac{99^3}{100^2}$ (c) $2+\dfrac{99^2}{100^3}$ (d) $3-\left(\dfrac{99}{100}\right)^2$

Ans. **(b)**

40. Consider a petrol pump which has a single petrol dispensing unit. Customers arrive there in accordance with a Poisson process having rate λ=1 minutes. An arriving customer enters the petrol pump only if there are two or less customers in the petrol pump, otherwise he/she leaves the petrol pump without taking the petrol (at any point of time a maximum of three customers are present in the petrol pump). Successive service times of the petrol dispensing unit are independent exponential random variables having mean 1/2 minutes. Let X denote the average number of customers in the petrol pump in the long run. Then $E(X)$ is equal to

(a) 7/15 (b) 3/5 (c) 11/15 (d) 13/15

Ans. **(c)**

41. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers. Let $b_n = \dfrac{a_n}{\max(a_1, ...a_n)}, n \geq 1$. Which of the following statements are necessarily true ?

(a) If $\lim_{n\to\infty} b_n$ exists in $\mathbb{R}$, then $\{a_n : n \geq 1\}$ is bounded

(b) If $\lim_{n\to\infty} b_n = 1$, then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$

(c) If $\lim_{n\to\infty} b_n = \frac{1}{2}$, then $\lim_{n\to\infty} a_n$ exists in $\mathbb{R}$

(d) If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$

Ans. **(c,d)**

42. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of real numbers. For $n \geq 1$ define

$$A_n = \begin{cases} a_n, & \text{if } a_n > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$B_n = \begin{cases} a_n, & \text{if } a_n < 0 \\ 0, & \text{otherwise} \end{cases}$$

Which of the following statements are necessarily true?

(a) $A_n \to 0$ and $B_n \to 0$ as $n \to \infty$

(b) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then both $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} B_n$, are absolutely convergent

(b) Both $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} B_n$ are convergent

(d) If $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent, then both $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} B_n$ are divergent

Ans. **(a,b,d)**

43. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x|x|$. Which of the following statements are true ?

(a) f is continuous on $\mathbb{R}$

(b) f is differentiable on $\mathbb{R}$

(c) f is differentiable only at 0

(d) f is not differentiable at 0

Ans. **(a,b)**

44. Let $(a_n)_{n\geq 1}$ be a bounded sequence of real numbers such that $\lim_{n\to\infty} a_n$ does not exist. Let $S = \{l \in \mathbb{R}$ there exists a subsequence of (a_n) converges to $l\}$. Which of the following statements are necessarily true?

(a) S is the empty set
(b) S has exactly one element
(c) S has at least two elements
(d) S has to be a finite set

Ans. **(c)**

45. Let $f:[0,1)\to[0,\infty)$ be defined by $f(x)=\dfrac{1}{1-x}$. For $n\geq 1$, let $p_n(x)=1+x+...+x^n$. Then which of the following statements are true?

(a) $f(x)$ is not uniformly continuous on $[0,1)$
(b) The sequence $(p_n(x))$ converges to $f(x)$ pointwise on $[0,1)$
(c) The sequence $(p_n(x))$ converges to $f(x)$ uniformly on $[0,1)$
(d) The sequence $(p_n(x))$ converges to $f(x)$ uniformly on $[0, c)$ for every $0 < c < 1$

Ans. **(a,b,d)**

46. Consider the improper integrals $I=\int_{\pi/2}^{\pi}\dfrac{1}{\sqrt{\sin x}}dx$ and, for $a\geq 0$ $I_a=\int_a^{\infty}\dfrac{1}{x\sqrt{1+x^2}}dx$

(a) The integral I is convergent
(b) The integral I is not convergent
(c) The integral I_a converges for $a=\dfrac{1}{2}$ but not for $a=0$
(d) The integral I_a converges for all $a\geq 0$

Ans. **(a,c)**

47. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous and one-to-one function. Which of the following statements are necessarily true?

(a) f is strictly increasing
(b) f is strictly decreasing
(c) f is either strictly increasing or strictly decreasing
(d) f is onto

Ans. **(c)**

48. Define by $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,y)=\begin{cases}\dfrac{y\sqrt{x^2+y^2}}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Which of the following statements are true ?

(a) $\dfrac{\partial f}{\partial x}(0,0)$ exists
(b) $\dfrac{\partial f}{\partial y}(0,0)$ exists
(c) f is not continuous at (0,0)
(d) f is not differentiable at (0,0)

Ans. **(a,b,c,d)**

49. Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ be a differentiable function such that $(Df)(0,0)$ has rank 2. Write $f = (f_1, f_2, f_3)$. Which of the following statements are necessarily true?

(a) f is injective in a neighbourhood of (0,0)

(b) There exists an open neighbourhood U of (0,0) in $\mathbb{R}^2$ such that f_3 is a function of f_1 and f_2

(c) f maps an open neighbourhood of (0,0) in $\mathbb{R}^2$ onto an open subset of $\mathbb{R}^3$

(d) (0,0) is an isolated point of $f^{-1}(\{f(0,0)\})$

Ans. (a,d)

50. Let $K \subseteq \mathbb{R}$ be non-empty and $f: K \to K$ be continuous such that $|x-y| \le |f(x) - f(y)|\ \forall x, y \in K$.

Which of the following statements are true ?

(a) f need not be surjective

(b) f must be surjective if $K = [0,1]$

(c) f is injective and $f^{-1}: f(K) \to K$ is continuous

(d) f is injective, but $f^{-1}: f(K) \to K$ need not be continuous

Ans. (a,b,c)

51. Let V be the subspace spanned by the vectors $v_1 = (1, 0, 2, 3, 1)$, $v_2 = (0, 0, 1, 3, 5)$, $v_3 = (0, 0, 0, 0, 1)$ in the real vector space $\mathbb{R}^5$. Which of the following vectors are in V?

(a) (1, 1, 1, 1, 1) (b) (0, 0, 1, 2, 4) (c) (1, 0, 1, 0, 1) (d) (1, 0, 1, 0, 2)

Ans. ( )

52. Consider $\mathbb{R}$ and $\mathbb{Q}[x]$ as vector spaces over $\mathbb{Q}$. Which of the following statements are true?

(a) There exists an injective $\mathbb{Q}$-linear transformation $T: \mathbb{R} \to \mathbb{Q}[x]$

(b) There exists an injective $\mathbb{Q}$-linear transformation $T: \mathbb{Q}[x] \to \mathbb{R}$

(c) The $\mathbb{Q}$-vector spaces $\mathbb{Q}[x]$ and $\mathbb{R}$ are isomorphic

(d) There do not exist non-zero $\mathbb{Q}$-linear transformations $T: \mathbb{R} \to \mathbb{Q}[x]$

Ans. ( )

53. Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear map with four distinct eigenvalues and satisfying $T^4 - 15T^2 + 10T + 24I = 0$. Which of the following statements are necessarily true ?

(a) There exists a non-zero vector $v_1 \in \mathbb{R}^4$ such that $Tv_1 = 2v_1$

(b) There exists a non-zero vector $v_2 \in \mathbb{R}^4$ such that $Tv_2 = v_2$

(c) For every non-zero vector $v \in \mathbb{R}^4$ the set $\{2v, 3Tv\}$ is linearly independent

(d) T is a one-one function

Ans. ( )

54. Let A be a 4×4 real matrix whose minimal polynomial is $x^2 + x + 1$ and let $B = A + I_4$. Which of the following statements are necessarily true?

(a) The minimal polynomial of B is $x^2 + x + 1$

(b) The minimal polynomial of B is $x^2 - x + 1$

(c) $B^3 = I_4$.

(d) $B^3 + I_4 = 0$

Ans. (b,d)

55. Let $V(\neq\{0\})$ be a finite dimensional vector space over $\mathbb{R}$ and $T:V\to V$ be a linear operator. Suppose that the kernel of T equals the image of T. Which of the following statements are necessarily true ?

(a) The dimension of V is even

(b) The trace of T is zero

(c) The minimal polynomial of T cannot have two distinct roots

(d) The minimal polynomial of T is equal to its characteristic polynomial

Ans. **(a,b,c)**

56. Let $M_5(\mathbb{C})$ be the complex vector space of 5×5matrices with entries in $\mathbb{C}$. Let V be a non-zero subspace of $M_5(\mathbb{C})$ such that every non-zero $A\in V$ is invertible. Which among the following are possible values for the dimension of V ?

(a) 1 (b) 2 (c) 3 (d) 5

Ans. **(a)**

57. Consider the real vector space $V=\mathbb{R}[x]$equipped with an inner product. Let W be the subspace of V consisting of polynomials of degree at most 2. Let $W^{\perp}$ denote the orthogonal complement of W in V. Which of the following statements are true?

(a) There exists a polynomial $p(x)\in W$ such that $x^4-p(x)\in W^{\perp}$

(b) $W^{\perp}=\{0\}$

(b) W and $W^{\perp}$ have the same dimension over $\mathbb{R}$

(d) $W^{\perp}$ is an infinite dimensional vector space over $\mathbb{R}$

Ans. **( )**

58. Let $q_1(x_1, x_2)$ and $q_2(y_1, y_2)$ be real quadratic forms such that there exist $(u_1, u_2),(v_1,v_2)\in\mathbb{R}^2$ such that $q(u_1, u_2)=1=q_2(v_1, v_2)$. Define $q(x_1, x_2, y_1, y_2) = q_1(x_1, x_2)-q_2(y_1, y_2)$.
Which of the following statements are necessarily true?

(a) q is a quadratic form in x_1, x_2, y_1, y_2

(b) There exists $(t_1,t_2)\in\mathbb{R}^2$ such that $q_1(t_1,t_2)=5$

(c) There does not exist $(s_1,s_2)\in\mathbb{R}^2$ such that $q_2(s_1,s_2)=-5$

(d) Given $\alpha\in\mathbb{R}$, there exists a vector $w\in\mathbb{R}^4$ such that $q(w)=\alpha$

Ans. **(a,b,d)**

59. Suppose that f is an entire function such that $|f(z)|\geq 2024$ for all $z\in\mathbb{C}$. Which of the following statements are necessarily true?

(a) $f(z)=2024$for all $z\in\mathbb{C}$ (b) f is a constant function

(c) f is an injective function (d) f is a bijective function

Ans. **(b)**

60. For $z \in \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}\sin\left(\frac{1}{z}\right)$ and $g(z) = f(z)\sin(z) = f(z)\sin(z)$. Which of the following statements are true?

(a) f has an essential singularity at 0

(b) g has an essential singularity at 0

(c) f has a removable singularity at 0

(d) g has a removable singularity at 0

Ans. **(a,b)**

61. Which of the following conditions ensure that the power series $\Sigma_{n\geq 0} a_n\, z^n$ defines an entire function?

(a) The power series converges for every $z \in \mathbb{C}$

(b) The power series converges for every $z \in \mathbb{R}$

(c) The power series converges for every $z \in \{2^n : n \in \mathbb{N}\}$

(d) The power series converges for every $z \in \left\{\frac{1}{5^n} : z \in \mathbb{N}\right\}$

Ans. **(a,b,c)**

62. Let f be an entire function such that for every integer $k \geq 1$there is an infinite set X_k, such that $f(z) = \frac{1}{k}$ for all $z \in X_k$. Which of the following statements are necessarily true?

(a) There exists an infinite set X such that $f(z) = 0$ for all $z \in X$

(b) There exists a non-empty closed set X such that $f(z) = 0$ for all $z \in X$

(c) The set X_k is unbounded for each $k \geq 1$

(d) If there exists a bounded sequence $(z_k)_{k\geq 1}$ such that $z_k \in X_k$, for each $k \geq 1$ then f has a zero

Ans. **(c,d)**

63. Let R be a principal ideal domain with a unique maximal ideal. Which of the following statements are necessarily true?

(a) Every quotient ring of R is a principal ideal domain

(b) There exists a quotient ring S of R and an ideal $I \subseteq S$ which is not principal

(c) R has countably many ideals

(d) Every quotient ring $S(\neq \{0\})$ of R has a unique maximal ideal which is principal

Ans. **(c,d)**

64. Let R and S be non-zero commutative rings with multiplicative identities 1_R, 1_S, respectively. Let $f : R \to S$ be a ring homomorphism with $(1_R) = 1_S$. Which of the following statements are true?

(a) If $f(a)$ is a unit in S for every non-zero element $a \in R$, , then S is a field

(b) If $f(a)$is a unit in S for every non-zero element $a \in R$, then $f(R)$ is a field

(c) If R is a field, then $f(a)$is a unit in S for every non-zero element $a \in R$

(d) If a is a unit in R, then $f(a)$ is a unit in S

Ans. **( )**

65. For two indeterminates x, y, let $R = \mathbb{F}_3[x]$ and $S = R[y]$. Which of the following statements are true?

(a) S is a principal ideal domain

(b) $S/(y^2 + x^2)$ is a unique factorization domain

(c) S is a unique factorization domain

(d) $S/(x)$ is a principal ideal domain

Ans. (c,d)

66. Which of the following numbers are order of some element of the symmetric group S_5?

(a) 3 (b) 4 (c) 5 (d) 6

Ans. (a,b,c,d)

67. Let I be an ideal of the ring $\mathbb{F}_2[t]/\left(t^2(1-t)^2\right)$. Which of the following are the possible values for the cardinality of I?

(a) 1 (b) 8 (c) 16 (d) 24

Ans. (a,b,c)

68. For which of the following values of q, does a finite field of order q have exactly 6 subfields?

(a) $q = 2^{18}$ (b) $q = 2^{32}$ (c) $q = 2^{12}$ (d) $q = 2^{243}$

Ans. (a)

69. Let X denote the topological space $\mathbb{R}$ with the cofinite topology (i.e., the finite complement topology) and let Y denote the topological space $\mathbb{R}$ with the Euclidean topology. Which of the following statements are true ?

(a) $X \times [0,1]$ is closed in $X \times Y$ with respect to the product topology

(b) $X \times [0,1]$ is compact with respect to the product topology

(c) X is compact

(d) $X \times Y$ is compact with respect to the product topology

Ans. (a,b,c)

70. Let τ be the smallest topology on the set $\mathbb{R}$ containing $\beta = \{[a,b) \mid a < b; a, b \in \mathbb{R}\}$. Which of the following statements are true?

(a) β is a basis for topology τ ?

(b) $\mathbb{R}$ is compact in the topology τ ?

(c) Topology τ is the same as the Euclidean topology

(d) Topology τ is Hausdorff

Ans. (a,d)

71. Consider the initial value problem (IVP): $y'(x) = \dfrac{\sin(y(x))}{1 + y^4(x)}, \quad x \in \mathbb{R}$. Then which of the following statements

$$y(0) = y_0$$

are true?

(a) There is a positive y_0 such that the solution of the IVP is unbounded

(b) There is a negative y_0 such that the solution of the IVP is bounded

(c) For every $y_0 \in \mathbb{R}$, every solution of the IVP is bounded

(d) For every $y_0 \in \mathbb{R}$, there is a solution to the IVP for all $x \in \mathbb{R}$

Ans. (b,c,d)

72. If $x_1 = x_1(t), x_2 = x_2(t)$ is the solution of the initial value problem $e^{-t}\dfrac{dx_1}{dt} = -x_1 + x_2$, $e^{-t}\dfrac{dx_2}{dt} = -x_1 - x_2$, $x_1(0) = 1, x_2(0) = 0$,and $r(t) = \sqrt{x_1^2(t) + x_2^2(t)}$, then which of the following statements are true?

(a) $r(t) \to 0$ as $t \to +\infty$

(b) $r(\ln 2) = e^{-1}$

(c) $r(\ln 2) = 2e^{-1}$

(d) $r(t)e^t \to 0\ as\ t \to +\infty$

Ans. **(a,b,d)**

73. Consider the boundary value problem (BVP) $\left(e^{-5x}y'\right)' + 6e^{-5x}y = -f(x), 0 < x < \ln 2,$

$$y(0) = 0,\ y(\ln 2) = 0$$

If $G(x,\xi) = \begin{cases} \left(e^{3x} + Be^{2x}\right)\left(Ce^{2\xi} + De^{3\xi}\right), & 0 \le \xi \le x, \\ \left(e^{3\xi} + Be^{2\xi}\right)\left(Ce^{2x} + De^{3x}\right), & x \le \xi \le \ln 2, \end{cases}$ (Green's function) is such

that $\int_0^{\ln 2} G(x,\xi)\, f(\xi)\, d\xi 2$ is the solution of the BVP, then the values of B, C and D are

(a) $B = -2, C = -1, D = 1$

(b) $B = -2, C = 1, D = -1$

(c) $B = 2, C = 1, D = 1$

(d) $B = 2, C = -1, D = -1$

Ans. **(b)**

74. . Let $B(0,2) = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\right\}$, and ∂B denote the boundary of B (0,2). Assume $(\alpha,\beta) \ne (0,0)\, m\, k \in \mathbb{R}$ and u is any solution to, $\begin{cases} -\Delta u = 0 & in\ B(0,2) \\ au(x,y) + \beta\dfrac{\partial u}{\partial v}(x,y) = 1 + \left(x^2 + y^2\right)k & on\ \partial B \end{cases}$

where $v(x, y)$ is the unit outward normal to B (0,2) at $(x,y) \in \partial B$. Consider the following statements:

S_1: If $\beta = 0$, then there exists a $(x_0, y_0) \in B(0,2)$ such that $\left|y(x_0,y_0)\right| = \dfrac{|1+4k|}{|\alpha|}$

S_2: If $\alpha = 0$ then $k = -\dfrac{1}{4}$. Then

(a) S_1 is true but S_2 is false

(b) S_2 is true but S_1 is false

(c) both S_1 and S_2 are true

(d) both S_1 and S_2 are false

Ans. **(c)**

75. Consider the initial boundary value problem (IBVP)

$$\begin{cases} u_t + u_x = 2u, & x > 0, t > 0 \\ u(0,t) = 1 + \sin t, & t > 0 \\ u(x,0) = e^x \cos x, & x > 0 \end{cases}$$

If u is the solution of the IBVP, then the value of $\dfrac{u(2\pi,\pi)}{u(\pi,2\pi)}$ is

(a) e^{π}

(b) $e^{-\pi}$

(c) $-e^{\pi}$

(d) $-e^{-\pi}$

Ans. **(c)**

76. Let S denote the set of all 2×2matrices A such that the iterative sequence generated by the Gauss-Seidel method applied to the system of linear equations $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ converges for every initial guess. Then which of the following statements are true ?

(a) $\begin{pmatrix} 5 & 8 \\ 1 & 2 \end{pmatrix}\in S$ (b) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\in S$ (c) $\begin{pmatrix} -3 & 1 \\ 2 & 3 \end{pmatrix}\in S$ (d) $\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}\in S$

Ans. (a,b,c)

77. Let $g(x)$ be the polynomial of degree at most 4 that interpolates the data

x	–1	0	2	3	6
y	–30	1	c	10	19

If $g(4) = 5$, then which of the following statements are true?

(a) $c=13$ (b) $g(5)=6$ (c) $g(1)=14$ (d) $c=15$

Ans. (b,c,d)

78. The extremizer of the problem $\min\left[\frac{1}{2}\int_{-1}^{1}\left[\left(y'(x)\right)^2+\left(y(x)\right)^2\right]dx\right]$ subject to $y\in C^1[-1,1], \int_{-1}^{1} xy(x)\,dx=0$ and $y(-1)=y(1)=1$ is

(a) $\frac{e}{1+e^2}\left(e^x+e^{-x}\right)+x^2-1$ (b) $\frac{e}{1+e^2}\left(e^x+e^{-x}\right)+1-x^2$

(c) $\frac{e}{1+e^2}\left(e^x+e^{-x}\right)$ (d) $\frac{e}{1+e^2}\left(e^x+e^{-x}\right)+\sin(2\pi x)$

Ans. (c)

79. The infimum of the set $\left\{\int_a^b\sqrt{1+\left(y'(t)\right)^2}\,dt : y\in C^1[a,b], y(a)=a^2, y(b)=b-5\right\}$ is

(a) $\frac{19\sqrt{2}}{8}$ (b) $19\sqrt{2}$ (c) $\frac{19}{8}$ (d) $\frac{19}{2\sqrt{2}}$

Ans. (a)

80. For $c\in\mathbb{R}$, consider the following Fredholm integral equation $y(x)=1+x+cx^2+2\int_0^1(1-3xt)\,y(t)\,dt$. Then the values of c for which the integral equation admits a solution are

(a) –8 (b) –6 (c) 2 (d) 6

Ans. (a)

81. For $\lambda\in\mathbb{R}$ such that $|\lambda|<\frac{5}{32}$, , let $R(x, t, \lambda)$ and u denote the resolvent kernel and the solution, respectively, of the Fredholm integral equation $u(x)+\frac{\lambda}{2}\int_{-2}^{2}\left(xt+x^2t^2\right)u(t)\,dt$. Then which of the following statements are true?

(a) $R(x,t,\lambda)=\frac{3xt}{3-8\lambda}-\frac{5x^2t^2}{5-32\lambda}$ (b) $R(x,t,\lambda)=\frac{3xt}{3-8\lambda}+\frac{5x^2t^2}{5-32\lambda}$

(c) $u(1)=-\frac{5}{5-32\lambda}$ (d) $u(1)=\frac{3}{3-8\lambda}$

Ans. (b,d)

82. Consider a solid torus of constant density ρ, formed by revolving the disc $(y-b)^2+z^2 \le a^2$, $x=0$ about the z-axis, where $0<a<b$. Then the moment of inertia of the solid torus about the z-axis is

(a) $2\pi^2 a^2 b^2\left(4b^2+3a^2\right)\rho$

(b) $\frac{\pi^2}{2}a^2 b\left(4b^2+3a^2\right)\rho$

(c) $\frac{\pi^2}{2}a^2 b\left(4a^2+3b^2\right)\rho$

(d) $2\pi^2 a^2 b^2\left(4a^2+3b^2\right)\rho$

Ans. **(b)**