CSIR-NET-(P YQ) MATHEMATICAL SCIENCE
JUNE-2017-I

PART-B

1. $L = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$. Then

(a) L = 0 (b) L = 1 (c) $0 < L < \infty$ (d) $L = \infty$

Ans. (a)

2. Consider the sequence

$a_n = \left(1 + (-1)^n \frac{1}{n}\right)^n$. Then

(a) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 1$ (b) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = e$

(c) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \frac{1}{e}$ (d) $\limsup_{n\to\infty} a_n = e, \liminf_{n\to\infty} a_n = \frac{1}{e}$

Ans. **(d)**

3. For $a > 0$, the series

$$\sum_{n=1}^{\infty} a^{\ln n}$$

is convergent if and only if

(a) $0 < a < e$ (b) $0 < a \le e$ (c) $0 < a < \frac{1}{e}$ (d) $0 < a \le \frac{1}{e}$

Ans. **(c)**

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then

(a) f is not continuous (b) f is continuous but not differentiable

(c) f is differentiable (d) f is not bounded

Ans. **(c)**

5. Let

A = $\{n \in \mathbb{N}$: n = 1 or the only prime factors of n are 2 or 3$\}$,

for example, $6 \in$ A, $10 \notin$ A.

Let $S=\Sigma_{n\in A}\dfrac{1}{n}$.Then

(a) A is finite
(b) S is a divergent series
(c) $S=3$
(d) $S=6$

Ans. **(c)**

6. For $n\geq 1$, let $f_n(x)=xe^{-nx^2}, x\in\mathbb{R}$.

Then the sequence $\{f_n\}$ is

(a) uniformly convergent on $\mathbb{R}$
(b) uniformly convergent only on compact subsets of $\mathbb{R}$
(c) bounded and not uniformly convergent on $\mathbb{R}$
(d) a sequence of unbounded functions

Ans. **(a)**

7. Let A be a 4×4 matrix. Suppose that the null space $N(A)$ of A is

$\{(x,y,z,w)\in\mathbb{R}^4 : x+y+z=0, x+y+w=0\}$. Then

(a) dim (column space(A)) = 1
(b) dim (column space(A)) = 2
(c) rank (A) = 1
(d) S={(1, 1, 1, 0), (1, 1, 0, 1)) is a basis of $N(A)$

Ans. **(b)**

8. Let A and B be real invertible matrices such that AB = – BA. Then

(a) Trace (A) = Trace (B) = 0
(b) Trace (A) = Trace (B) =1
(c) Trace (A) = 0, Trace (B) = 1
(d) Trace (A) = 1, Trace (B) = 0

Ans. **(a)**

9. Let A be an $n\times n$ self-adjoint matrix with eigenvalues $\lambda_1,....\lambda_n$.

Let $\|X\|_2=\sqrt{|x_1|^2+.....+|x_n|^2}$ for $X=(x_1,....,x_n)\in\mathbb{C}^n$

If $p(A)=a_0I+a_1A+....+a_nA^n$ then $\sup\|X\|_2=1\|p(A)X\|_2$ is equal to

(a) $\max\{a_0+a_1\lambda_j+....+a_n\lambda_j^n : 1\leq j\leq n\}$

(b) $\max\{|a_0+a_1\lambda_j+....+a_n\lambda_j^n| : 1\leq j\leq n\}$

(c) $\min\{a_0+a_1\lambda_j+....+a_n\lambda_j^n : 1\leq j\leq n\}$

(d) $\min\{|a_0+a_1\lambda_j+....+a_n\lambda_j^n| : 1\leq j\leq n\}$

Ans. **(b)**

10. Let $p(x) = ax^2 + \beta x + \gamma$ be a polynomial, where $\alpha,\beta,\gamma\in\mathbb{R}$. Fix $x_0\in\mathbb{R}$.

Let $S=\{(a,b,c)\in\mathbb{R}^3 : p(x)=a(x-x_0)^2+b(x-x_0)+c \text{ for all } x\in\mathbb{R}\}$

Then the number of elements in S is

(a) 0 (b) 1

(c) strictly greater than 1 but finite (d) infinite

Ans. (b)

11. Let $A=\begin{bmatrix}1 & 0 & 2\\ 1 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}$ and I be the 3×3 identity matrix.

If $6A^{-1} = aA^2 + bA + cI$ for $a, b, c\in\mathbb{R}$ then (a, b, c) equals

(a) (1, 2, 1) (b) (1, –1, 2) (c) (4, 1, 1) (d) (1, 4, 1)

Ans. (d)

12. Let $A=\begin{bmatrix}1 & 1 & 2\\ 1 & -2 & 5\\ 2 & 5 & -3\end{bmatrix}$. Then the eigenvalues of A are

(a) $-4, 3, -3$ (b) $4, 3, 1$ (c) $4, -4\pm\sqrt{13}$ (d) $4, -2\pm2\sqrt{7}$

Ans. (c)

13. Let C denote the unit circle centered at the origin in $\mathbb{C}$.

Then $\dfrac{1}{2\pi i}\int_C \left|1+z+z^2\right|^2 dz,$

where the integral is taken anti-clockwise along C, equals

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (c)

14. Consider the power series

$$f(x)=\sum_{n=2}^{\infty}\log(n)\,x^n$$

The radius of convergence of the series $f(x)$ is

(a) 0 (b) 1 (c) 3 (d) ∞

Ans. (b)

15. For an odd integer $k\ge 1$, let F be the set of all entire function f such that

$f(x)=\left|x^k\right|$ for all $x\in(-1,1)$. Then the cardinality of F is

(a) 0 (b) 1

(c) strictly greater than 1 but finite (d) infinite

Ans. (a)

16. Suppose f is holomorphic in an open neighbourhood of $z_0\in\mathbb{C}$. Given that the series

$$\sum_{n=0}^{\infty} f^{(n)}(z_0)$$

converges absolutely, we can conclude that

(a) f is constant (b) f is a polynomial

(c) f can be extended to an entire function (d) $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$

Ans. **(c)**

17. Let S be the set of all integers from 100 to 999 which are neither divisible by 3 nor divisible by 5. The number of elements in S is

(a) 480 (b) 420 (c) 360 (d) 240

Ans. **(a)**

18. The remainder obtained when 16^{2016} is divided by 9 equals

(a) 1 (b) 2 (c) 3 (d) 7

Ans. **(a)**

19. Consider the ideal $I = (x^2 + 1, y)$ in the polynomial ring $\mathbb{C}[x, y]$. Which of the following statements is true?

(a) I is a maximal ideal

(b) I is a prime ideal but not a maximal ideal

(c) I is a maximal ideal but not a prime ideal

(d) I is neither a prime ideal nor a maximal ideal

Ans. **(d)**

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Choose the correct statement

(a) f is bounded

(b) The image of f is an open subset of $\mathbb{R}$

(c) $f(A)$ is bounded for all bounded subsets A of $\mathbb{R}$

(d) $f^{-1}(A)$ is compact for all compact subsets A of $\mathbb{R}$

Ans. **(c)**

21. Suppose $x: [0, \infty) \to [0, \infty)$ is continuous and $x(0) = 0$. If $\left(x(t)\right)^2 \le 2 + \int_0^t x(s)\,ds, \quad \forall t \ge 0$. Then which of the following is TRUE ?

(a) $x\left(\sqrt{2}\right) \in [0, 2]$ (b) $x\left(\sqrt{2}\right) \in \left[0, \frac{3}{\sqrt{2}}\right]$

(c) $x\left(\sqrt{2}\right) \in \left[\frac{5}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right]$ (d) $x\left(\sqrt{2}\right) \in [10, \infty)$

Ans. **(b)**

22. The solution of the partial differential equation $u_t - xu_x + 1 - u = 0, \; x \in \mathbb{R}, t > 0$ subject to $u(x, 0) = g(x)$ is

(a) $u(x,t) = 1 - e^{-t}\left(1 - g\left(xe^t\right)\right)$ (b) $u(x,t) = 1 + e^{t}\left(1 - g\left(xe^t\right)\right)$

(c) $u(x,t) = 1 - e^{-t}\left(1 - g\left(xe^{-t}\right)\right)$ (d) $u(x,t) = e^{-t}\left(1 - g\left(xe^t\right)\right)$

Ans. **(*)**

23. Suppose $u \in C^2(B)$, B is the unit ball in $\mathbb{R}^2$, satisfies

$\Delta u = f$ in B

$$au + \frac{\partial u}{\partial n} = g \text{ on } \alpha > 0$$

where n is the unit outward normal to B. If a solution exists then

(a) it is unique
(b) there are exactly two solutions
(c) there are exactly three solutions.
(d) there are infinitely many solutions

Ans. (a)

24. The magnitude of the truncation error for the scheme $f'(x) = Af(x) + Bf(x+h) + Cf(x+2h)$ is equal to

(a) $h^2 f'''(\xi)$ if $A = -\frac{5}{6h}, B = \frac{3}{2h}, C = -\frac{2}{3h}$
(b) $h^2 f'''(\xi)$ if $A = \frac{5}{6h}, B = \frac{3}{2h}, C = \frac{2}{3h}$
(c) $h^2 f''(x)$ if $A = -\frac{5}{6h}, B = \frac{3}{2h}, C = -\frac{2}{3h}$
(d) $h^2 f''(x)$ if $A = \frac{5}{6h}, B = \frac{3}{2h}, C = \frac{2}{3h}$

Ans. (*)

25. The infimum of $\int_0^1 (u'(t))^2\, dt$ on the class of functions $\{u \in C^1[0,1]$ such that $u(0) = 0$ and $\max_{[0,1]} |u| = 1\}$ is equal to

(a) 0
(b) 1/2
(c) 1
(d) 2

Ans. (c)

26. Let $f(x)$ be the solution of $\int_0^x e^{x-t}\phi(t)\,dt = x, \;\; x > 0$. Then $\phi(1)$ equals

(a) –1
(b) 0
(c) 1
(d) 2

Ans. (b)

27. A rigid body having one point O fixed and no external torque about O has equal principal moments of inertia. Then the body must rotate with

(a) angular velocity of variable magnitude
(b) angular velocity with constant magnitude
(c) constant angular momentum but varying angular velocity
(d) varying angular momentum with varying angular velocity

Ans. (b)

28. Consider a spherical pendulum consisting of a particle of mass m which moves under gravity on a smooth sphere of radius a. In terms of spherical polar angles θ, ϕ, with θ measured up from the downward vertical, the Lagrangian is given by

(a) $ma\left[\frac{a}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - g\cos\theta\right]$
(b) $ma\left[\frac{a}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + g\cos\theta\right]$
(c) $ma\left[\frac{a}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \cos^2\theta\right) + g\sin\theta\right]$
(d) $ma\left[\frac{a}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \cos^2\theta\right) - g\sin\theta\right]$

Ans. (b)

29. A box contains 40 numbered red balls and 60 numbered black balls. From the box, balls are drawn one by one at random without replacement till all the balls are drawn. The probability that the last ball drawn is black equals

(a) 1/100
(b) 1/60
(c) 3/5
(d) 2/3

Ans. (c)

30. $X_1, X_2, \ldots$ are independent identically distributed random variables having common density f. Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct

(a) $\frac{1}{n}(X_1 + \ldots + X_n) \to 0$ in probability

(b) $\frac{1}{n}(X_1+...+X_n)\to 0$ almost surely

(c) $P\left(\frac{1}{\sqrt{n}}(X_1+....+X_n)<0\right)\to\frac{1}{2}$

(d) $\Sigma_{i=1}^n X_i$ has the same distribution as $\Sigma_{i=1}^n(-1)^i X_i$

Ans. **(d)**

31. Let N_t, denote the number of accidents up to time t. Assume that $\{N_t\}$ is a Poisson process with intensity 2. Given that there are exactly 5 accidents during the time period [20, 30], what is the conditional probability that there is exactly one accident during the time period [15,25]?

(a) $\frac{15}{32}e^{-10}$ (b) $20e^{-20}$ (c) $\frac{10^5}{5!}e^{-30}$ (d) $\frac{1}{5}$

Ans. **(a)**

32. X and Y are independent random variables each having the density

$$f(t)=\frac{1}{\pi}\frac{1}{1+t^2},-\infty<t<\infty.$$

Then the density function of $\frac{X+Y}{3}$ for $-\infty<t<\infty$ is given by

(a) $\frac{6}{\pi}\frac{1}{4+9t^2}$ (b) $\frac{6}{\pi}\frac{1}{9+4t^2}$ (c) $\frac{3}{\pi}\frac{1}{1+9t^2}$ (d) $\frac{3}{\pi}\frac{1}{9+t^2}$

Ans. **(a)**

33. Suppose { X1,.... Xn }, $n\geq 2$, is a random sample from the distribution with probability density function

$$f(x;\theta)=\begin{cases}\frac{\theta^\theta}{\Gamma(\theta)}x^{\theta-1}e^{-x\theta} & ;x>0\\ 0 & ;x\leq 0\end{cases}$$

with $\theta>0$. Then the method of moments estimator of 0

(a) does not exist (b) is $\frac{n}{\Sigma_{i=1}^n(x_i-1)^2}$ (c) is $\frac{n}{\Sigma_{i=1}^n(X_i-\bar{X})^2}$ (d) is $\frac{n-1}{\Sigma_{i=1}^n(X_i-1)^2}$

Ans. **(c)**

34. Let $X_1, X_2,..., X_n$ for $n\geq 5$ be a random sample from the distribution with probability density function

$$f(x;\theta)=\begin{cases}e^{-(x-\theta)} & \text{if } x>\theta\\ 0 & \text{otherwise}\end{cases}$$

for $\theta>0$. The confidence coefficient of the confidence interval

$$\left[\min\{X_1,....,X_n\}-\frac{\ln 4}{n}, \min\{X_1,...X_n\}+\frac{\ln 2}{n}\right] \text{ for } \theta, \text{ is}$$

(a) 0.5 (b) 0.75 (c) 0.95 (d) $1-\frac{1}{2^n}$

Ans. **(b)**

55. Let X be a random sample from an exponential distribution with mean $1/\lambda$. If λ a has a prior distribution with probability density function

$$g(\lambda)=\begin{cases}\lambda e^{-\lambda} & ;\lambda>0\\ 0 & \lambda\le 0\end{cases}$$

then the Bayes estimator of 1/λ with respect to the squared error loss function is

(a) $\frac{2}{X+1}$ (b) $\frac{1}{X}$ (c) X (d) $\frac{X+1}{2}$

Ans. **(d)**

56. Consider the linear statistical model

$$y_{ij}=\mu+\tau_i+\varepsilon_{ij};\ i=1,2,....,a;\ j=1,2,....,n$$

where μ is unknown, τ_i are independently and identically distributed as N(0,σ_i^2), independently and identically ε_{ij} are distributed as N(0,σ^2) τ_i, and τ_{ij} are independent for all *i* and *j*. Note that τ_i is the i^{th} treatment effect. Suppose SS_{total}, $SS_{treatment}$, SS_{error} are total sum of squares, total treatment sum of squares and error sum of squares, respectively.

To test : $H_0:\sigma_i^2=0$ vs $H_A:\sigma_\tau^2>0$ which of the following statements is not true ?

(a) The sum of squares identity is SS_{total}+ $SS_{treatment}$+ SS_{error}

(b) $SS_{error}\sim\sigma^2x^2n(a-1)$

(c) under $H_0,\dfrac{\frac{SS_{treatment}}{a-1}}{\frac{SS_{error}}{n(a-1)}}\sim F_{a-1,n(a-1)}$

(d) $E(SS_{error})=n(a-1)(\sigma^2+n\sigma_\tau^2)$

Ans. **(b) or (c) or (d)**

57. Suppose (X1, X2) follows a bivariate normal distribution with $E(X_1)=E(X_2)=0$,

$V(X_1)=V(X_2)=2$ and $Cov(X_1,X_2)=-1$. If $\phi(x)=\frac{1}{\sqrt{2x}}\int_{-\infty}^{x}e^{-y^2j^2}dy$ then $P[X_1-X_2>6]$ is equal to

(a) $\phi(-1)$ (b) $\phi(-3)$ (c) $\phi(\sqrt{6})$ (d) $\phi(-\sqrt{6})$

Ans. **(d)**

58. Consider the problem of drawing a sample of size 2 from a finite population of size 20. The sampling is done with replacement using probability proportional to size sampling scheme. The normed size measures P_1,....,P_{20} are given by $p_i=\frac{1}{40},i=1,....10,\ p_i=\frac{3}{40},\ i=11,...,20$. The expected number of distinct units drawn is

(a) $\frac{83}{80}$ (b) $\frac{157}{80}$ (c) $\frac{17}{16}$ (d) $\frac{31}{16}$

Ans. **(d)**

59. If we interchange two columns of a Latin square design (LSD), then the new design is

(a) an LSD

(b) a completely randomised design (CRD) but not an LSD

(c) a randomised block design (RBD) but not an LSD

(d) a balanced incomplete block design (BIBD) but not an LSD

Ans. (a)

60. Consider the LPPMinimize $c^t x$ subject to $Ax = b,\ x \geq 0$, where $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ 0 & -1 & -2 & -3 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$c = (2,-1,1,-9,0)^t$ and $x = (X_1, X_2, X_3, X_4, X_5)'$. Using the revised simplex method with current basis as $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, which of the following statements is correct?

(a) The next entering variable is x_5

(b) The solution corresponding to the current basis is optimal

(c) The next entering variable is x_4

(d) The next entering variable is x_3

Ans. (c)

PART-C

61. Let $\alpha = 0.10110111011110...$ be a given real number written in base 10, that is, the n-th digit of α is 1, unless n is of the form $\frac{k(k+1)}{2} - 1$ in which case it is 0. Choose all the correct statements from below

(a) α is a rational number

(b) α is an irrational number

(c) For every integer $q \geq 2$, there exists an integer $r \geq 1$ such that $\frac{r}{q} < \alpha < \frac{r+1}{q}$

(d) α has no periodic decimal expansion

Ans. (b, d)

62. For $a, b \in \mathbb{N}$, consider the sequence

$$d_n = \frac{\binom{n}{a}}{\binom{n}{b}}$$

for $n > a, b$. Which of the following statements are true? As $n \to \infty$,

(a) $\{d_n\}$ converges for all values of a and b

(b) $\{d_n\}$ converges if $a < b$

(c) $\{d_n\}$ converges if $a = b$

(d) $\{d_n\}$ converges if $a > b$

Ans. (b, c)

63. Let $\{a_n\}$ be a sequence of real numbers satisfying $\Sigma_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$. Then the series $\Sigma_{n=0}^{\infty} a_n x^n, x \in \mathbb{R}$ is convergent

(a) nowhere on $\mathbb{R}$

(b) everywhere on $\mathbb{R}$

(c) on some set containing (–1, 1)

(d) only on (–1,1)

Ans. (c)

64. Let $f(x) = \tan^{-1} x, x \in \mathbb{R}$ Then

(a) there exists a polynomial $p(x)$ satisfying $p(x)\, f'(x) = 1$, for all x

(b) $f^{(n)}(0) = 0$ for all positive even integers n

(c) the sequence $\{f^{(n)}(0)\}$ is unbounded

(d) $f^{(n)}(0) = 0$ for all n

Ans. (a), b , c)

65. Let $f_n(x) = \dfrac{1}{1+n^2x^2}$ for $n \in \mathbb{N}, x \in \mathbb{R}$ .Which of the following are true ?

(a) f_n converges pointwise on [0, 1] to a continuous function

(b) f_n converges uniformly on [0, 1]

(c) f_n converges uniformly on $\left[\frac{1}{2}, 1\right]$

(d) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \left(\lim\limits_{n\to\infty} f_n(x)\right) dx$

Ans. (c , d)

66. If $\lambda_n = \int_0^1 \dfrac{dt}{(1+t)^n}$ for $n \in \mathbb{N}$, then

(a) λ_n does not exist for some n

(b) λ_n exists for every n and the sequence is unbounded

(c) λ_n exists for every n and the sequence is bounded

(d) $\lim\limits_{n\to\infty} (\lambda_n)^{1/n} = 1$

Ans. (c, d)

77. The equation $11^x + 13^x + 17^x - 19x = 0$ has

(a) no real root

(b) only one real root

(c) exactly two real roots

(d) more than two real roots

Ans. (b)

78. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$f(\underline{x}) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$, where $\underline{x} = (x_1, x_2, \ldots\ldots, x_n)$ and at least one a_j is not zero. Then we can conclude that

(a) f is not everywhere differentiable

(b) the gradient $(\nabla f)(\underline{x}) \neq 0$ for every $\underline{x} \in \mathbb{R}^n$

(c) if $\underline{x} \in \mathbb{R}^n$ is such that $(\nabla f)(\underline{x}) = 0$ then $f(\underline{x}) = 0$

(d) if $\underline{x} \in \mathbb{R}^n$ is such that $f(\underline{x}) = 0$ then $(\nabla f)(\underline{x}) = 0$

Ans. (c)

79. Let S be the set of $(\alpha, \beta)\in\mathbb{R}^2$ such that

$$\frac{x^\alpha y^\beta}{\sqrt{x^2+y^2}} \to 0 \text{ as} \left(x, y\right) \to \left(0,0\right)$$

Then S is contained in

(a) $\{(\alpha, \beta): \alpha > 0, \beta > 0\}$ (b) $\{(\alpha, \beta): \alpha > 2, \beta > 2\}$

(c) $\{(\alpha, \beta): \alpha + \beta > 1\}$ (d) $\{(\alpha, \beta): \alpha + 4\beta > 1\}$

Ans. (a, c, d)

70. Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers $a_0, a_1, ..., a_k$. For $p \in V$ $\max\left\{\left|p\left(a_j\right)\right| : 0 \le j \le k\right\}$ defines a norm on V

(a) only if $k < n$ (b) only if $k \ge n$ (c) if $k+1 \le n$ (d) if $k \ge n+1$

Ans. (b ,d)

71. Let V be the vector space of polynomials of degree at most 3 in a variable x with coefficients in $\mathbb{R}$. Let $T = d/dx$ be the linear transformation of V to itself given by differentiation. Which of the following are correct?

(a) T is invertible

(b) 0 is an eigenvalue of T

(c) There is a basis with respect to which the matrix of T is nilpotent.

(d) The matrix of T with respect to the basis $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is diagonal

Ans. (b, c)

72. Let m, n, r be natural numbers. Let A be an $m\times n$ matrix with real entries such that $(AA^t)^r = I$, where I is the $m\times m$ identity matrix and A^t is the transpose of the matrix A. We can conclude that

(a) $m = n$

(b) AA^t is invertible

(c) $A^t A$ is invertible

(d) if $m = n$, then A is invertible

Ans. (b, d)

73. Let A be an $n\times n$ real matrix with $A^2 = A$. Then

(a) the eigenvalues of A are either 0 or 1

(b) A is a diagonal matrix with diagonal entries 0 or 1

(c) rank (A) = trace (A)

(d) rank $(I - \text{A})$ = trace $(1 - \text{A})$

Ans. (a, c, d)

74. For any $n\times n$ matrix B, let $N(B) = \{X\in\mathbb{R}^n : BX = 0\}$ be the null space of B. Let A be a 4×4 matrix with $\dim(N(A-2I)) = 2$, $\dim(N(A-4I)) = 1$ and rank $(A) = 3$. Then

(a) 0, 2 and 4 are eigenvalues of A (b) determinant $(A) = 0$

(c) A is not diagonalizable (d) trace (A) = 8

Ans. (a , b , d)

75. Which of the following 3×3 matrices are diagonalizable over $\mathbb{R}$?

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Ans. (a, c)

76. Let H be a real Hilbert space and $M \subseteq H$ be a closed linear subspace. Let $x_0 \in H \setminus M$. Let $y_0 \in M$ be such that

$\|x_0 - y_0\| = \inf\{\|x_0 - y\| : y \in M\}$

Then

(a) such a y_0 is unique (b) $x_0 \perp M$ (c) $y_0 \perp M$ (d) $x_0 - y_0 \perp M$

Ans. (a , d)

77. Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $Q(X) = X^t AX$ for $X \in \mathbb{R}^3$. Then

(a) A has exactly two positive eigenvalues

(b) all the eigenvalues of A are positive

(c) $Q(X) \geq 0$ for all $X \in \mathbb{R}^3$

(d) $Q(X) < 0$ for some $X \in \mathbb{R}^3$

Ans. (a, d)

78. Consider the matrix

$$A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}; x \in \mathbb{R}. \text{ Then}$$

(a) $A(x)$ has eigenvalue 0 for some $x \in \mathbb{R}$

(b) 0 is not an eigenvalue of $A(x)$ for any $x \in \mathbb{R}$

(c) $A(x)$ has eigenvalue 0 for all $x \in \mathbb{R}$

(d) $A(x)$ is invertible for every $x \in \mathbb{R}$

Ans. (a)

79. Let $f = u + iv$ be an entire function where u, v are the real and imaginary parts of f respectively. If the Jacobian matrix

$$J_a = \begin{bmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{bmatrix}$$

is symmetric for all $a \in \mathbb{C}$. then

(a) f is a polynomial

(b) f is a polynomial of degree ≤ 1

(c) f is necessarily a constant function

(d) f is a polynomial of degree strictly greater than 1.

Ans. (a, b)

80. Consider the function $f(z) = \dfrac{\sin(\pi z/2)}{\sin(\pi z)}$, Then f has poles at

(a) all integers (b) all even integers

(c) all odd integers (d) all integers of the form $4k + 1$, $k \in \mathbb{Z}$

Ans. (c , d)

11. Consider the Mobius transformation $f(z) = \frac{1}{z}, z \in \mathbb{C}, z \neq 0,$. If C denotes a circle with positive radius passing through the origin, then f maps C\{0} to

(a) a circle.
(b) a line.
(c) a line passing through the origin.
(d) a line not passing through the origin.

Ans. **(b, d)**

82. For which among the following functions $f(z)$ defined on G = $\mathbb{C}\backslash\{0\}$, is there no sequence of polynomials approximating $f(z)$ uniformly on compact subsets of G?

(a) $\exp(z)$ (b) $1/z$ (c) z^2 (d) $1/z^2$

Ans. **(b , d)**

83. For an integer $n \geq 2$, let S_n be the permutation group on n letters and A_n, the alternating group. Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication. Which of the following are correct statements ?

(a) For every integer $n \geq 2$, there is a nontrivial homomorphism $\chi: S_n \to \mathbb{C}^*$.
(b) For every integer $n \geq 2$, there is a unique nontrivial homomorphism $\chi: S_n \to \mathbb{C}^*$.
(c) For every integer $n \geq 3$, there is a nontrivial homomorphism $\chi: A_n \to \mathbb{C}^*$
(d) For every integer n $2 \geq 5$, there is no nontrivial homomorphism $\chi: A_n \to \mathbb{C}^*$.

Ans. **(a, b, d)**

84. Let $R = \{ f : \{1, 2, ..., 10\} \to \mathbb{Z}_2 \}$ be the set of all $\mathbb{Z}_2$-valued functions on the set {1,2,..., 10} of the first ten positive integers. Then R is commutative ring with pointwise addition and pointwise multiplication of functions. Which of the following statements are correct?

(a) R has a unique maximal ideal
(b) Every prime ideal of R is also maximal
(c) Number of proper ideals of R is 511
(d) Every element of R is idempotent

Ans. **(b , d)**

85. Which of the following rings are principal ideal domains (PID) ?

(a) $\mathbb{Q}[x]$ (b) $\mathbb{Z}[x]$ (c) $(\mathbb{Z}/6\mathbb{Z})[x]$ (d) $(\mathbb{Z}/7\mathbb{Z})[x]$

Ans. **(a, d)**

86. Let G be a group of order 125. Which of the following statements are necessarily true ?

(a) G has a non-trivial abelian subgroup
(b) The centre of G is a proper subgroup
(c) The centre of G has order 5
(d) There is a subgroup of order 25

Ans. **(a, d)**

87. Let R be a non-zero ring with identity such that $a^2 = a$ for all $a \in \mathbb{R}$. Which of the following statements are true?

(a) There is no such ring
(b) $2a = 0$ for all $a \in \mathbb{R}$
(c) $3a = 0$ for all $a \in \mathbb{R}$
(d) $\mathbb{Z}/2\mathbb{Z}$ is a subring of $\mathbb{R}$

Ans. **(b, d)**

88. Which of the following polynomials are irreducible in $\mathbb{Z}[x]$?

(a) $x^4 + 10x + 5$ (b) $x^3 - 2x + 1$ (c) $x^4 + x^2 + 1$ (d) $x^3 + x + 1$

Ans. **(a, d)**

89. Let X be any topological space. Let $A \subseteq X$ be nonempty. For $x, y \in A$, define $x \sim y$ if there is a connected subset $C \subseteq A$ such that $x, y \in C$. For $x \in A$, define $C(x) = \{y \in A : y \sim x\}$. Then

(a) $C(x) = C(y) \Rightarrow x = y$
(b) $C(x) = C(y) \Rightarrow x \sim y$
(c) $C(x) \cap C(y) \neq \phi \Rightarrow x \sim y$
(d) $C(x) \cap C(y) \neq \phi \Rightarrow C(x) = C(y)$

Ans. **(b, c, d)**

90. Let X be a topological space and Y a subset of X. Write i: $Y \to X$ for the inclusion map.

Choose the correct statement(s):

(a) If Y has the subspace topology, then i is continuous

(b) If i is continuous, then Y has the subspace topology

(c) If Y is an open subset of X, then $i(U)$ is open in X for all subsets $U \subseteq Y$ that are open in the subspace topology on Y

(d) If Y is a compact subset of X, then $i(U)$ is open in X for all subsets $U \subseteq Y$ that are open in the subspace topology on Y

Ans. (a, c)

91. Consider the solution of the ordinary differential equation $y'(t) = -y^3 + y^2 + y$ subject to $y(0) = y_0 \in (0, 2)$.

Then $\lim_{t\to\infty} y(t)$ belongs to

(a) $\{-1, 0\}$ (b) $\{-1, 2\}$ (c) $\{0, 2\}$ (d) $\{0, +\infty\}$

Ans. (b, c)

92. If the solution to

$$\begin{cases} \dfrac{du}{dx} = y^2 + x^2 & , x > 0 \\ y(0) = 2 \end{cases}$$

exists in the interval $[0, L_0)$ and the maximal interval of existence of

$$\begin{cases} \dfrac{dz}{dx} = z^2 & , x > 0 \\ z(0) = 1 \end{cases}$$

is $[0, L_1)$, then which of the following statements are correct

(a) $L_1 = 1, L_0 > 1$ (b) $L_1 = 1, L_0 \leq 1$ (c) $L_1 < 2, L_0 \leq 1$ (d) $L_1 > 2, L_0 < 1$

Ans. (*)

93. Consider the partial differential equation

$x\dfrac{\partial u}{\partial x} + yu\dfrac{\partial u}{\partial y} = -xy$ for $x > 0$ subject to $u = 5$ on $xy = 1$. Then

(a) $u(x,y)$ exists when $xy \leq 19$ and $u(x,y) = u(y, x)$ for $x > 0, y > 0$

(b) $u(x, y)$ exists when $xy \geq 19$ and $u(x, y) = u(y, x)$ for $x > 0, y > 0$

(c) $u(1,11) = 3, u(13,-1) = 7$

(d) $u(1, -1) = 5, u(11, 1) = -5$

Ans. (a, c)

94. If a complete integral of the partial differential equation

$$x\left(p^2 + q^2\right) = zp;\ p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$$

passes through the curve $x = 0, z^2 = 4y$, then the envelope of this family passing through $x = 1$ and $y = 1$ has

(a) $z = -2$ (b) $z = 2$ (c) $z = \sqrt{2 + 2\sqrt{2}}$ (d) $z = -\sqrt{2 + 2\sqrt{2}}$

Ans. (c, d)

95. For a differentiable function $f:\mathbb{R}\to\mathbb{R}$ define the difference quotient

$$(D_x f)(h)=\frac{f(x+h)-f(x)}{h};\ h>0$$

Consider numbers of the form $\hat{h}=h(1+\in)$ for a fixed $\in>0$ and let

$$e_1(h)=f'(x)-(D_x f)(h),$$

$$e_2(h)=(D_x f)(h)-(D_x f)(\hat{h})$$

$$e(h)=e_1(h)+e_2(h)$$

if $f(x+h)=f(x+h)$, then

(a) $e_1(h)\to 0$ as $h\to 0$

(b) $e_2(h)\to 0$ as $h\to 0$

(c) $e_2(h)\to\in f'(x)/(1+\in)$ as $h\to 0$

(d) $e(h)\to 0$ as $h\to 0$

Ans. (a, c)

96. Let y_n satisfy $y_n=y_{n-1}+hy_{n-1}$ with $y_0=1(n=1,2,...,N)$ and for $0<h<1,\ Nh=1$. Then

(a) $y_N\to e$ as $N\to\infty$

(b) $y_N\to e^h$ as $N\to\infty$

(c) $y_n=(1+h)^n$

(d) $y_n\geq 1$

Ans. (a, c, d)

97. Let $y(x)$ be the solution of the integral equation $y(x)=x-\int_0^x xt^2y(t)\,dt,\ x>0$. Then the value of the function $y(x)$ at $x=\sqrt{2}$ is equal to

(a) $\frac{1}{\sqrt{2e}}$ (b) $\frac{e}{2}$ (c) $\frac{\sqrt{2}}{e^2}$ (d) $\frac{\sqrt{2}}{e}$

Ans. (d)

98. The solutions for $\lambda=-1$ and $\lambda=3$ of the integral equation $y(x)=1+\lambda\int_0^1 K(x,t)y(t)\,dt,$

where $K(x,t)=\begin{cases}\cosh x\sinh t & ,0\leq x\leq t\\ \cosh t\sinh x & ,t\leq x\leq 1\end{cases}$ are, respectively,

(a) $-\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cos 2x}{\cos 2-2\sin 2\tanh 1}+1\right)$

(b) $-\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cosh 2x}{\cosh 2-2\sinh 2\tanh 1}+1\right)$

(c) $\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cosh 2x}{\cosh 2-2\sinh 2\tanh 1}-1\right)$

(d) $\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cos 2x}{\cos 2-2\sin 2\tanh 1}-1\right)$

Ans. (b)

99. Consider the functional $I(y(x))=\int_{x_0}^{x_1} f(x,y)\sqrt{1+y'^2}e^{\tan^{-1}y'}\,dx$ where $f(x,y)\neq 0$. Let the left end of the extremal be fixed at the point $A(x_0, y_0)$ and the right end $B(x_1,y_1)$ be movable along the curve $y=\psi(x)$. Then the extremal $y=y(x)$ intersects the curve $y=\psi(x)$ along which the boundary point $B(x_1, y_1)$ slides at an angle

(a) $\pi/3$ (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

Ans. **(c)**

100. Let B be the unit ball in $\mathbb{R}^2$. Let $u\in C^2(\overline{B})$ be a minimizer of

$$I(u)=\int_B\left(|\nabla u|^2+fu\right)dx+\int_{\partial B} au^2 ds$$

where f and a are continuous functions in $C^2(\overline{B})$. Let $\vec{n}$ denote the unit outward normal.

Which of the following are correct?

(a) $-2\Delta u+f=0$ in B and $\frac{\partial u}{\partial \vec{n}}+au=0$ on ∂B

(b) $-2\Delta u+f+a=0$ in B and $\frac{\partial u}{\partial \vec{n}}=0$ on ∂B

(c) $-\Delta u+f+a=0$ in B and $2\frac{\partial u}{\partial \vec{n}}+au=0$ on ∂B

(d) $-\Delta u+2f=0$ in B and $2\frac{\partial u}{\partial \vec{n}}+au=0$ on ∂B

Ans. **(a)**

101. . Let q_α and p_α ($\alpha=1,2,\ldots..,n$) be the generalized coordinates and the generalized momenta, respectively. If H denotes the Hamiltonian and q_α (for some $\alpha=\alpha_0$) is an ignorable coordinate, then which of the following equations are satisfied ?

(a) $\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}, \dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha}, \forall\alpha$

(b) $\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}, \dot{q}_\alpha=-\frac{\partial H}{\partial p_\alpha}\ \forall\alpha$

(c) $\dot{p}_{\alpha_0}=0,\ \dot{q}_{\alpha_0}=\frac{\partial H}{\partial p_{\alpha_0}}$

(d) $\dot{p}_{\alpha_0}=-\frac{\partial H}{\partial q_{\alpha_0}},\ \dot{q}_{\alpha_0}=0$

Ans. **(a, c)**

102. For a conservative system, the end configurations are fixed and the velocity in the varied motion is such that $T+V=E$. Here T,V and E represent, respectively the kinetic energy, the potential energy and the total energy . If δ (A) denotes the infinitesimal change in a variable A, and p_α and q_α ($\alpha=1,2,\ldots,n$) represent the generalized momenta and generalized coordinates, respectively, then

(a) $\delta\int T dt=0$

(b) $\delta\int\sum_{\alpha=1}^{n} p_\alpha dq_\alpha=0$

(c) $\delta\int\sum_{\alpha=1}^{n} q_\alpha dp_\alpha=0$

(d) $\delta\int\sum_{\alpha=1}^{n}\left(q_\alpha dp_\alpha+q_\alpha dp_\alpha\right)=0$

Ans. **(a, b)**