

Chapter 7

Angular Momentum and Torque

Angular momentum: ($\vec{L}$)

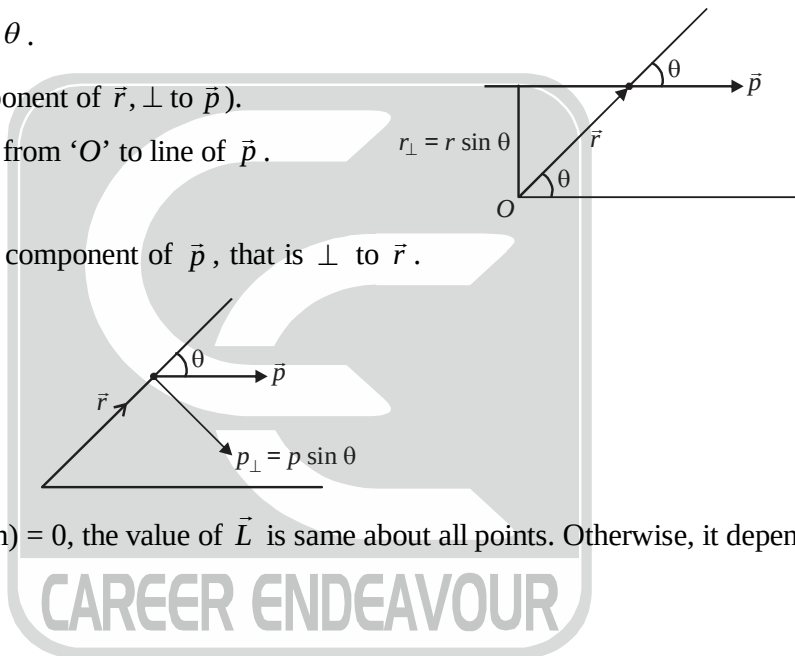
$\vec{L}$ of any particle moving with linear momentum $\vec{p}$ at a point $\vec{r}$, w.r.t. an origin (point of reference 'O') is $\vec{L} = \vec{r} \times \vec{p} \Rightarrow |\vec{L}| = L = pr \sin \theta$.

$L = pr_{\perp}$, (where $r_{\perp}$ is the component of $\vec{r}$, $\perp$ to $\vec{p}$).

It is the perpendicular distance from 'O' to line of $\vec{p}$.

$$\Rightarrow L = rp \sin \theta$$

$\Rightarrow L = r p_{\perp}$, where $p_{\perp}$ is the component of $\vec{p}$, that is $\perp$ to $\vec{r}$.



If $\sum_i \vec{p}_i$ (Net linear momentum) = 0, the value of $\vec{L}$ is same about all points. Otherwise, it depends on origin chosen.

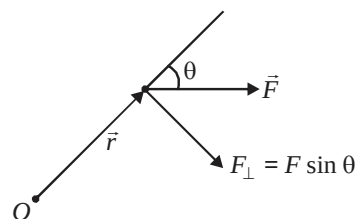
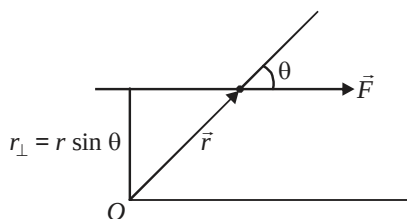
Torque: ($\vec{\tau}$)

Angular momentum is defined as: $\vec{L} = \vec{r} \times \vec{p}$

$$\begin{aligned} \therefore \frac{d}{dt}(\vec{L}) &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= (\vec{v} \times m\vec{v}) + \vec{r} \times \vec{F} \end{aligned}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

This ($\vec{\tau}$) is called as the torque of $\vec{F}$ on the particle at $\vec{r}$ w.r.t. an origin 'O'.



$|\vec{\tau}| = \tau = Fr \sin \theta = Fr_{\perp} = rF_{\perp}$; where $r_{\perp}$ and $F_{\perp}$ are components of $\vec{r}$ and $\vec{F}$, perpendicular to $\vec{F}$ and $\vec{r}$, respectively.

If $\sum_i \vec{F}_i = 0$, the value of $\vec{\tau}$ about any point is same.

$\vec{L}$ or $\vec{\tau}$ about an axis:

The angular momentum of a particle about an axis AB is given by

$$L_{AB} = \hat{n} \cdot \vec{L} = \hat{n} \cdot (\vec{r} \times \vec{p}); \text{ where } \vec{L} \text{ is found using any point on the axis.}$$

If $\vec{\tau} = 0$ for any system.

$$\Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$$

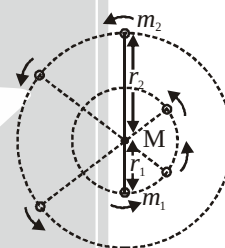
Double star problem: A double star consists of two stars revolving in circular orbits about their common centre of mass. The necessary centripetal force for circular motion is provided by their mutual gravitational attraction. Let us consider a double star which consists of two stars of masses m_1 and m_2 separated by a distance r .

There is no external force on the system therefore centre of mass of the system does not move.

Let r_1 and r_2 be radii of the orbits of the two stars.

$$\therefore r_1 = \frac{m_2 r}{m_1 + m_2} \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1},$$

$$r_2 = \frac{m_1 r}{m_1 + m_2}$$



Since centre of mass is stationary. Therefore, the angular velocity of the two stars must be same.

$$\therefore \text{For star 1 } m_1 \omega^2 r_1 = \frac{Gm_1 m_2}{r^2} \Rightarrow \omega = \sqrt{\frac{G(m_1 + m_2)}{r^3}}$$

$$\text{For star 2 } m_2 \omega^2 r_2 = \frac{Gm_1 m_2}{r^2} \Rightarrow \omega = \sqrt{\frac{G(m_1 + m_2)}{r^3}}$$

Ratio of linear speeds of two stars:

$$\frac{v_1}{v_2} = \frac{\omega r_1}{\omega r_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1} \quad \therefore \frac{v_1}{v_2} = \frac{m_2}{m_1}$$

Ratio of kinetic energy of two stars

$$\frac{K_1}{K_2} = \frac{\frac{1}{2} m_1 v_1^2}{\frac{1}{2} m_2 v_2^2} = \frac{m_1 (\omega r_1)^2}{m_2 (\omega r_2)^2} = \frac{m_1}{m_2} \cdot \left(\frac{r_1}{r_2}\right)^2 = \frac{m_2}{m_1} \quad \therefore \frac{K_1}{K_2} = \frac{m_2}{m_1}$$

Ratio of angular momentum of two stars

$$\frac{L_1}{L_2} = \frac{m_1 v_1 r_1}{m_2 v_2 r_2} = \frac{m_1 \omega r_1^2}{m_2 \omega r_2^2} = \frac{m_2}{m_1} \quad \therefore \frac{L_1}{L_2} = \frac{m_2}{m_1}$$

$$\text{Ratio of magnitude of linear momentum } \frac{P_1}{P_2} = \frac{m_1 v_1}{m_2 v_2} = \frac{m_1 \omega r_1}{m_2 \omega r_2} = \frac{m_1}{m_2} \cdot \frac{r_1}{r_2} = \frac{m_1}{m_2} \cdot \frac{m_2}{m_1} \quad \therefore \frac{P_1}{P_2} = 1$$

$$\text{Total energy of double star system } E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1.m_2}{r} = \frac{1}{2}m_1\omega^2r_1^2 + \frac{1}{2}m_2\omega^2r_2^2 - \frac{Gm_1.m_2}{r}$$

put values of r_1, r_2 and ω to get

$$E = \frac{1}{2} \frac{G(m_1 + m_2)}{r^3} \left[\frac{m_1m_2^2r^2 + m_1^2r^2m_2}{(m_1 + m_2)^2} \right] - \frac{Gm_1m_2}{r} = \frac{Gm_1.m_2}{2r} - \frac{Gm_1.m_2}{r} \Rightarrow E = \frac{-Gm_1.m_2}{2r}$$

SOLVED PROBLEMS

1. Three particles are moving in x-y plane with constant velocities as shown in the figure. Calculate angular momentum of each particle about the origin.

Soln. Angular momentum can be calculated in various ways.

Let us use $L = r_{\perp} p$

For 1 kg particle:

$$\text{momentum } (p) = mv = 1 \times 3 = 3 \text{ kg m/s}$$

perpendicular distance of line of momentum from origin ($r_{\perp}$) = 2m

$$\therefore L = r_{\perp} p = 2 \times 3 = 6 \text{ kg m}^2/\text{s}$$

For 2kg particle: Line of momentum of this particle passes through origin therefore $r_{\perp} = 0 \therefore L = 0$

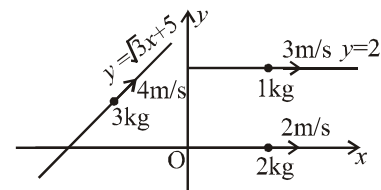
The perpendicular distance of a straight line $ax + by + c = 0$ from a point (x', y') is equal to

$$p = \frac{|ax' + by' + c|}{\sqrt{a^2 + b^2}}$$

$$\text{For 3kg particle: } r_{\perp} = \frac{|y - \sqrt{3}x - 5|_{x=0, y=0}}{\sqrt{1^2 + (\sqrt{3})^2}} = \frac{|0 - \sqrt{3} \times 0 - 5|}{\sqrt{4}} = \frac{5}{2}$$

Its momentum is given by

$$p = mv = 3 \times 4 = 12 \text{ kg m/s} \quad \therefore L = r_{\perp} p = \frac{5}{2} \times 12 = 30 \text{ kg m}^2/\text{sec}$$



2. A particle of mass m is thrown with initial speed u at angle α with horizontal. If maximum height reached is much small compared to radius of earth. Calculate angular momentum of particle about point of projection after time t .

Soln. Position vector of particle after time t

$$\vec{r} = x\hat{i} + y\hat{j}$$

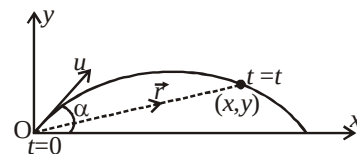
$$\Rightarrow \vec{r} = (u \cos \alpha)t\hat{i} + \left[(u \sin \alpha)t - \frac{1}{2}gt^2 \right]\hat{j}$$

velocity of particle after time t

$$\vec{v} = \frac{d\vec{r}}{dt} = u \cos \alpha \hat{i} + (u \sin \alpha - gt)\hat{j}$$

The angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (u \cos \alpha)t & (u \sin \alpha)t - \frac{1}{2}gt^2 & 0 \\ u \cos \alpha & u \sin \alpha - gt & 0 \end{vmatrix}$$



$$= \hat{k} \left[u^2 \cos \alpha (\sin \alpha) t - gt^2 u \cos \alpha - u^2 \cos \alpha (\sin \alpha) t + \frac{1}{2} gu (\cos \alpha) t^2 \right] = -\frac{1}{2} gt^2 u \cos \alpha \hat{k}$$

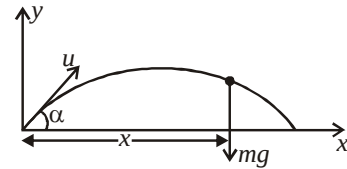
Another method: At time t , torque on the particle about point of projection is

$$\tau = r_{\perp} F = xmg, \text{ or } \vec{\tau} = -mg \times \hat{k} = -mg u (\cos \alpha) t \hat{k}$$

We know that $\vec{\tau} = \frac{d\vec{L}}{dt}$

$$\therefore \frac{d\vec{L}}{dt} = -mg u (\cos \alpha) t \hat{k} \quad \text{or} \quad \int_0^{\vec{L}} d\vec{L} = -mg u \cos \alpha \int_0^t t dt \hat{k}$$

$$\therefore \vec{L} = -\frac{1}{2} mg u t^2 \cos \alpha \hat{k}$$



3. A particle of mass ' m ' moves with constant speed v along a circle of radius R . Calculate angular momentum about centre of circle. Also calculate angular momentum about a point on the circumference of circle.

Soln. When it is making an angle θ w.r.t. origin, the angular momentum about centre is $L_c = r_{\perp} \cdot p = Rmv$

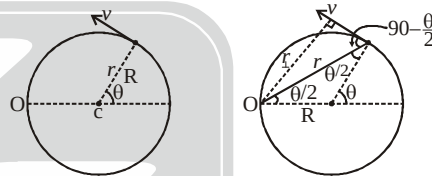
About a point on circumference

$$L_O = r_{\perp} p = r \sin \left(90 - \frac{\theta}{2} \right) mv \quad (\text{from figure})$$

$$= r \cos \frac{\theta}{2} mv = \left(2R \cos \frac{\theta}{2} \right) \cos \frac{\theta}{2} mv$$

$$(\text{Use sine rule to get } \frac{r}{\sin(\pi - \theta)} = \frac{R}{\sin \theta/2} \Rightarrow r = 2R \cos \frac{\theta}{2})$$

$$= 2Rmv \cos^2 \frac{\theta}{2}$$



4. A particle of mass m moving with constant speed v strikes a floor elastically. Calculate change in angular momentum about a point which is at a distance ' d ' from the point of impact as shown in figure.

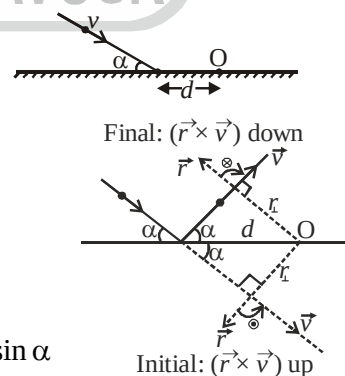
Soln. The collision is elastic therefore particle will rebound with same speed and at same angle with the floor as shown in the figure.

$$\therefore r_{\perp} = d \sin \alpha$$

Angular momentum before impact

$$L_1 = r_{\perp} p = d \sin \alpha mv, \quad L_2 = r_{\perp} p = -d \sin \alpha mv$$

$$\therefore \text{Change in angular momentum } \Delta L = L_2 - L_1 = -2mvd \sin \alpha$$

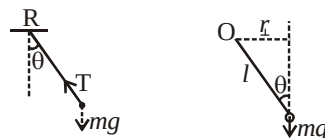


5. A simple pendulum consisting of a small bob of mass m and light string of length ' l ' oscillates in vertical plane. Calculate torque due to gravity and tension in the string at the instant string makes θ angle with downward vertical.

Soln. Line of action of tension passes through

point of suspension. Therefore $r_{\perp} = 0$

Thus, torque due to tension = 0



For mg , $r_{\perp} = l \sin \theta$

$$\therefore \tau = r_{\perp} F = l \sin \theta mg = mgl \sin \theta$$

6. An L-shaped uniform wire is suspended from a string as shown in figure. If length of the two sections of wire be 'a' and 'b' then what is value of angle θ made by upper section with vertical.

Soln. The wire is in rotational equilibrium therefore centre of mass of the rod must lie on the vertical line through point A so that gravity does not produce any torque about point A. To find the position of centre of mass let us choose coordinate system as shown in figure.

Let λ be mass per unit length of the wire.

$$\therefore X_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\lambda a \cdot \frac{a}{2} + \lambda b a}{\lambda a + \lambda b} = \frac{a(a + 2b)}{2(a + b)}$$

$$Y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{\lambda a \cdot 0 + \lambda b \cdot \frac{b}{2}}{\lambda a + \lambda b} = \frac{b^2}{2(a + b)}$$

$$\therefore \tan \theta = \frac{Y_{CM}}{X_{CM}} = \frac{b^2}{a(a + 2b)}$$

$$\text{or } \theta = \tan^{-1} \left(\frac{b^2}{a(a + 2b)} \right) \text{ if } a = b \text{ then } \theta = \tan^{-1} \left(\frac{1}{3} \right)$$

7. A solid hemisphere of mass M and radius R is placed against a smooth wall as shown in the figure. What is normal reaction on the sphere due to the wall.

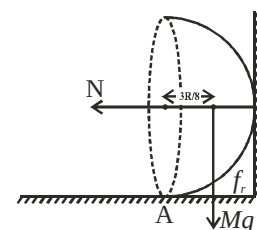
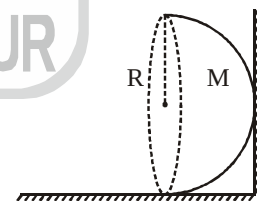
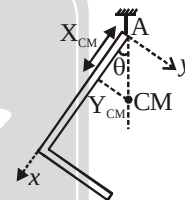
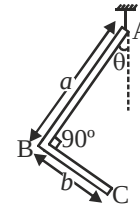
Soln: Centre of mass of the hemisphere lies at a distance $\frac{3R}{8}$ from

its base centre. The sphere is in equilibrium therefore torque about point A must be zero.

$$\therefore NR - Mg \cdot \frac{3R}{8} = 0$$

$$\text{or, } N = \frac{3Mg}{8}$$

8. A thin rod of mass M and length L is suspended from one end while its other end lies on a smooth horizontal plane and makes an angle α with the plane. Find the normal reaction from the plane and tension in the string from which it is suspended.



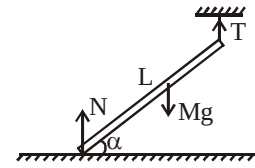
Soln. For translational equilibrium $T + N = Mg$... (a)

For rotational equilibrium

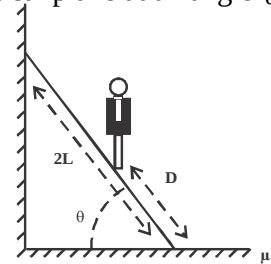
Torque about lower end of the rod is zero

$$\therefore T \cdot L \cos \alpha - Mg \frac{L}{2} \cos \alpha = 0 \quad \therefore T = \frac{Mg}{2}$$

$$\text{from (a) } N = Mg - T = Mg - \frac{Mg}{2} = \frac{Mg}{2}$$



9. A uniform ladder of length $2L$ and mass 'm' leans against a wall in a vertical plane at an angle θ to the horizontal. The floor is rough, having a coefficient of static friction μ . A person of mass M stands on the ladder at a distance D from its base (see figure). If the wall is frictionless, the maximum distance ($D_{\max}$) up the ladder that the person can reach before the ladder slips is



Soln. Let us take ladder plus person as system. Various forces acting on the system are shown in the figure. For translational equilibrium we must have

$$N_1 = (M + m)g \quad \dots (a)$$

$$N_2 = f_r \quad \dots (b)$$

For rotational equilibrium torque about lower end of the ladder must be zero. Therefore,

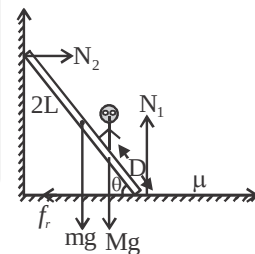
$$MgD \cos \theta + mgL \cos \theta - N_2 \cdot 2L \sin \theta = 0$$

$$\therefore N_2 = \frac{g \cot \theta}{2} \left[\frac{MD}{L} + m \right]$$

$$\therefore \text{from (b) we get, } f_r = \frac{g \cot \theta}{2} \left[\frac{D}{L} M + m \right]$$

$$\text{since } f_r \leq \mu N_1, \quad \therefore \frac{g \cot \theta}{2} \left[\frac{D}{L} M + m \right] \leq \mu (M + m)g$$

$$\text{or } D \leq \frac{L}{M} \left[2\mu(M + m) \tan \theta - m \right] \quad \therefore D_{\max} = L \left[2\mu \left(1 + \frac{m}{M} \right) \tan \theta - \frac{m}{M} \right]$$



10. Angular momentum of an object about a point varies with time as $\vec{L} = \alpha \hat{i} + \beta t^2 \hat{j}$. At a certain instant of time, torque acting on the object about that point makes 60° with $\vec{L}$. Calculate magnitude of torque at that instant.

Soln. Torque is related to angular momentum as

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \vec{\tau} = \frac{d}{dt} (\alpha \hat{i} + \beta t^2 \hat{j}) \Rightarrow \vec{\tau} = 2\beta t \hat{j} \quad \dots (i)$$

$$\text{given, } \theta = \cos^{-1} \left[\frac{|\vec{L} \cdot \vec{\tau}|}{|\vec{L}| |\vec{\tau}|} \right] = 60^\circ \Rightarrow \cos^{-1} \left[\frac{2\beta^2 t^3}{2\beta t (\alpha^2 + \beta^2 t^4)^{1/2}} \right] = 60^\circ$$

$$\text{or } \frac{2\beta^2 t^3}{2\beta t(\alpha^2 + \beta^2 t^4)^{1/2}} = \frac{1}{2}$$

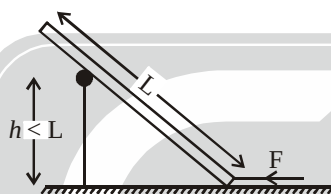
$$\text{or } \beta^2 t^4 = \frac{1}{4}(\alpha^2 + \beta^2 t^4) \text{ or } 4\beta^2 t^4 - \beta^2 t^4 - \alpha^2 = 0$$

$$3\beta^2 t^4 = \alpha^2 \text{ or } t = \left[\frac{\alpha^2}{3\beta^2} \right]^{1/4}$$

Therefore, from (i), the torque at this instant will be

$$\text{Therefore, } \tau = 2\beta \left[\frac{\alpha^2}{3\beta^2} \right]^{1/4} \hat{j}$$

11. Consider the situation shown in the figure. All the surfaces are smooth and the rod has mass M .



What is the minimum value of F required to keep the rod in equilibrium?

Soln. Clearly, the force will be minimum when upper end of the rod lies on the round support. We are excluding the case when rod is vertical.

$$\text{In figure, } \sin \theta = \frac{h}{L} \Rightarrow \tan \theta = \frac{h}{\sqrt{L^2 - h^2}}$$

For translational equilibrium

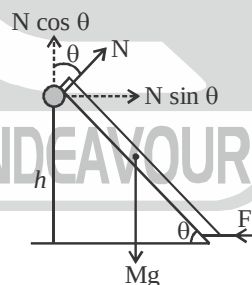
$$N \cos \theta = Mg$$

$$N \sin \theta = F$$

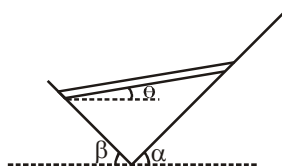
$$\therefore \tan \theta = \frac{F}{Mg}$$

$$\text{or } F = Mg \tan \theta$$

$$F = Mg \cdot \frac{h}{\sqrt{L^2 - h^2}}$$



12. A thin rod of mass M and length L is lying on two inclined smooth planes having angles of inclination α and β as shown in the figure. Calculate the angle θ that the rod makes with horizontal in equilibrium.



Soln. The forces acting on the rod are shown in the figure.
In equilibrium,

$$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma \tau = 0$$

From $\Sigma F_x = 0$, we get

$$N_1 \sin \beta = N_2 \sin \alpha$$

$$\text{or } N_1 = \frac{N_2 \sin \alpha}{\sin \beta}$$

From $\Sigma F_y = 0$, we get

$$N_1 \cos \beta + N_2 \cos \alpha = Mg$$

using (1) we get

$$\frac{N_2 \sin \alpha \cdot \cos \beta}{\sin \beta} + N_2 \cos \alpha = Mg$$

$$\text{or } N_2 = \frac{Mg \cdot \sin \beta}{\sin(\alpha + \beta)} \quad \dots (ii)$$

Now, from $\Sigma \tau_A = 0$, we get

$$\vec{AB} \times \vec{N}_2 + \vec{AC} \times (-Mg\hat{j}) = 0 \quad \dots (iii)$$

Here $\vec{AB} = L(\cos \theta \hat{i} + \sin \theta \hat{j})$

$$\vec{N}_2 = N_2(-\sin \alpha \hat{i} + \cos \alpha \hat{j})$$

$$\vec{AC} = \frac{L}{2}(\cos \theta \hat{i} + \sin \theta \hat{j})$$

$\therefore$ from (iii) we get

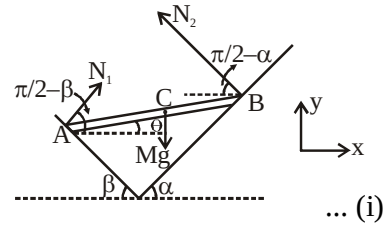
$$LN_2(\cos \alpha \cos \theta + \sin \theta \sin \alpha) \hat{k} - \frac{MgL}{2} \cos \theta \hat{k} = 0$$

$$\text{Using } N_2, \text{ we get, } \frac{MgL \sin \beta}{\sin(\alpha + \beta)} (\cos \alpha \cos \theta + \sin \alpha \sin \theta) - \frac{MgL}{2} \cos \theta = 0$$

$$\text{or } \left[\frac{1}{2} - \frac{\sin \beta \cos \alpha}{\sin(\alpha + \beta)} \right] \cos \theta - \frac{\sin \alpha \cdot \sin \beta}{\sin(\alpha + \beta)} \sin \theta = 0$$

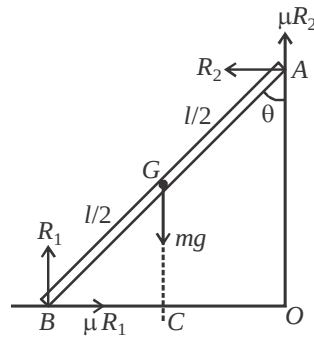
$$\text{or } \tan \theta = \frac{2 \sin \beta \cdot \cos \alpha - \sin(\alpha + \beta)}{2 \sin \alpha \cdot \sin \beta} = \frac{1}{2}(\cot \beta - \cot \alpha)$$

$$\therefore \theta = \tan^{-1} \left[\frac{1}{2}(\cot \beta - \cot \alpha) \right]$$



13. A uniform ladder AB of length 3 meters and weight 20 kg rests with the end A against rough vertical wall and the end B on level ground. If the wall and the ground are equally rough and the coefficient of friction is 0.5, find the limiting position of equilibrium.

Soln. The different forces on the ladder are shown in figure. Let in the limiting position of equilibrium, the ladder makes an angle θ with the wall.



Equating the horizontal and vertical forces, we get

$$R_2 = \mu R_1 \quad \dots (i)$$

$$R_1 + \mu R_2 = mg \quad \dots (ii)$$

From equation (i) & (ii),

$$R_1 + \mu^2 R_1 = mg \quad \text{or} \quad R_1 = \frac{mg}{(1 + \mu^2)} \quad \dots (iii)$$

From equation (i),

$$R_2 = \left(\frac{\mu}{1 + \mu^2} \right) mg \quad \dots (iv)$$

Taking the moment of forces about B, we get

$$mg \times BC = R_2 \times OA + \mu R_2 \times OB.$$

From figure,

$$OA = l \cos \theta, \quad OB = l \sin \theta \quad \text{and} \quad BC = \frac{l}{2} \sin \theta$$

$$\therefore \quad mg \frac{l}{2} \sin \theta = R_2 l \cos \theta + \mu R_2 l \sin \theta$$

$$\Rightarrow \quad \sin \theta \left[\frac{mg}{2} - \mu R_2 \right] = R_2 \cos \theta$$

$$\Rightarrow \quad \tan \theta = \frac{R_2}{\left[\frac{mg}{2} - \mu R_2 \right]} = \frac{\left(\frac{\mu}{1 + \mu^2} \right) mg}{\left[\frac{mg}{2} - \frac{\mu^2}{1 + \mu^2} mg \right]} = \frac{\mu / (1 + \mu^2)}{(1 + \mu^2 - 2\mu^2) / (2 + 2\mu^2)}$$

$$\Rightarrow \quad \tan \theta = \frac{2\mu}{1 - \mu^2} = \frac{2 \times 0.5}{1 - (0.5)^2} = 1.33 \quad (\because \mu = 0.5)$$

$$= \tan^{-1}(1.33)$$

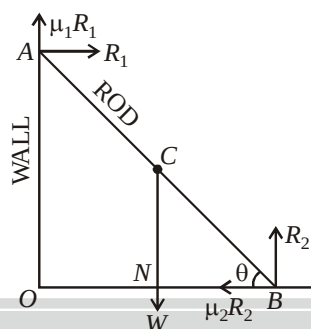
$$\theta = 53^\circ 4'.$$

14. A uniform rod is made to lean between a rough vertical wall and the ground. Show that the least angle at which the rod can be leaned without slipping is given by

$$\theta = \tan^{-1} \left(\frac{1 - \mu_1 \mu_2}{2\mu_2} \right),$$

where μ_1 and μ_2 stand for the coefficient of friction between (i) the rod and the wall, and (ii) the rod and the ground.

Soln. The situation is shown in figure.



Considering the equilibrium of the rod, we have

$$R_1 = \mu_2 R_2 \quad \dots (i)$$

$$\text{and } R_2 + \mu_1 R_1 = W \quad \dots (ii)$$

Taking the moments of the forces about A, we have

$$R_2 \times OB = W \times ON + \mu_2 R_2 \times OA$$

$$\Rightarrow R_2 \times AB \cos \theta = W \times \frac{AB \cos \theta}{2} + \mu_2 R_2 \times AB \sin \theta$$

$$\Rightarrow \cos \theta \left(R_2 - \frac{W}{2} \right) = \sin \theta (\mu_2 R_2).$$

$$\therefore \tan \theta = \frac{\left(R_2 - \frac{W}{2} \right)}{\mu_2 R_2} \quad \dots (iii)$$

From equations (i) and (ii), we have

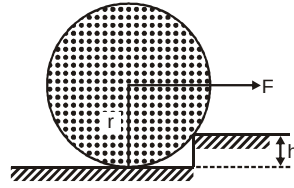
$$W = R_2 + \mu_1 R_1 = R_2 + \mu_1 \mu_2 R_2 = R_2 (1 + \mu_1 \mu_2)$$

$$\text{Now } R_2 - \frac{W}{2} = R_2 - \frac{R_2}{2} [(1 + \mu_1 \mu_2)] = R_2 \left(1 - \frac{1}{2} - \frac{\mu_1 \mu_2}{2} \right) = \frac{R_2}{2} (1 - \mu_1 \mu_2).$$

$$\therefore \tan \theta = \frac{\frac{R_2}{2} (1 - \mu_1 \mu_2)}{\mu_2 R_2} = \frac{1 - \mu_1 \mu_2}{2\mu_2}$$

$$\therefore \theta = \tan^{-1} \left(\frac{1 - \mu_1 \mu_2}{2\mu_2} \right).$$

15. A uniform solid wheel of mass M and radius r is halted at a step of height h as shown in the figure. The minimum force F , applied horizontally at the centre of the wheel, necessary to raise the wheel over this step is



[TIFR 2016]

- (a) $Mg \frac{\sqrt{h(2r-h)}}{r+h}$ (b) $Mg \frac{\sqrt{h(2r+h)}}{r-h}$ (c) $Mg \frac{\sqrt{h(r+h)}}{r-h}$ (d) $Mg \frac{\sqrt{h(2r-h)}}{r-h}$

Soln. Since the wheel is about to climb and contact is lost, $N = 0$

In equilibrium, $\sum \tau_i = 0, \sum F_i = 0$

Torque about 'O' $\sum \tau_0 = 0$

$$\Rightarrow \vec{r}_1 \times M\vec{g} + \vec{r}_1 \times \vec{F} = 0$$

$$\Rightarrow \vec{r}_1 \times [M\vec{g} + \vec{F}] = 0$$

$$\Rightarrow rMg \sin(180^\circ - \theta) - rF \sin(90^\circ + \theta) = 0$$

$$\Rightarrow F = Mg \tan \theta = Mg \frac{\sqrt{r^2 - (r-h)^2}}{r-h}$$

$$F = \frac{Mg\sqrt{2hr-h^2}}{r-h} = \frac{Mg\sqrt{h(2r-h)}}{(r-h)}$$

Correct option is (d)

